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Mathematical Reviews

Edited by

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R. P. Boas, Jr., *Executive Editor*

Vol. 11, No. 7 July-August, 1950 pp. 487–570

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MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
THE INSTITUTE OF MATHEMATICAL STATISTICS
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Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence 12, R. I.

Subscriptions: Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions may be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence 12, Rhode Island, but should preferably be addressed to the American Mathematical Society, 531 West 116th Street, New York 27, N. Y.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Air Research, Department of the Air Force, U. S. A. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in paragraph (d-2), section 34.40, P. L. and R. of 1948, authorized November 9, 1940.

Mathematical Reviews

Vol. 11, No. 7 JULY-AUGUST, 1950 Pages 487-570

FOUNDATIONS

*Frege, G. **Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl. The Foundations of Arithmetic. A logico-mathematical enquiry into the concept of number.** English translation by J. L. Austin. Philosophical Library, New York, N. Y., 1950. xii+xii+XII+XII+119+119 pp. $4.75.

The German original was published in Breslau in 1884. In this edition the German text and English translation are on facing pages.

*Curry, Haskell B. **A Theory of Formal Deducibility.** Notre Dame Mathematical Lectures, no. 6. University of Notre Dame, Notre Dame, Ind., 1950. ix+126 pp.

Chapter I treats the methodology of formal systems. A formal system is defined by a system of objects called terms, a set of propositions concerning the terms, a set of propositions called axioms and rules of derivation. A terminology is introduced for these concepts, comprising nouns for the terms, predicators for the predicates which form propositions from terms, etc. This terminology, the *A*-language of the system, must be adjoined to the *U*-language, i.e., the communicative language understood by speaker and hearer. To connect this conception with the ideas of Hilbert and his school, we have to consider the propositions as terms of a new system with one predicator $\vdash$. [This seems, indeed, an adequate description of Hilbert's underlying ideas which he never expressed systematically; many of his followers interpreted his system as referring to the signs as terms; the author ascribes this "syntactical" approach to Carnap.]

The following three chapters contain an exposition of symbolic logic treated by Gentzen's methods. Four systems are considered: the minimal *M*, the intuitionistic *J*, the classical *K* and the system *D*, which bears somewhat the same relation to *K* that *M* does to *J*; e.g., Glivenko's theorem can be extended as follows: A necessary and sufficient condition that *A* is derivable in *D* (*K*) is that $\neg\neg A$ is derivable in *M* (*J*). For each of the four systems a complete set of rules in Gentzen's *L*-calculus as well as in his *N*-calculus is given; also a complete set of axioms (calculus *HM*, *HJ*, *HK*, *HD*). A thorough knowledge of the fundamental ideas of formal logic and of the general theorems concerning it can be obtained from the book, rather than skill in actual derivations.

Chapter V contains a discussion of modalities from an original point of view. Given a formal system *S* we may distinguish those theorems which are derivable from only a part of its rules, constituting a system $S^{\#}$ within *S*. Then we can interpret $\#A$ ("*A* is necessary") to mean that *A* is derivable in $S^{\#}$. We are thus led to the rules (in the *N*-calculus):

$$\text{(Ye)}\quad \frac{\#A}{A}, \qquad \text{(Yi)}\quad \frac{\overset{(\#)}{A}}{\#A},$$

where (#) over (Yi) indicates that *A* is derived by the rules of $S^{\#}$. Strict implication is defined by $A \dashv B. = .\#(A \supset B)$. The system obtained in this way is equivalent to S4 of Lewis.

A. Heyting (Amsterdam).

Henkin, Leon. Fragments of the propositional calculus. J. Symbolic Logic **14**, 42–48 (1949).

The author shows how one can obtain a complete set of axioms (based on the rule of detachment) for any fragment of the classical sentential calculus which contains material implication. He considers the question whether the given sets of axioms are always independent, provides a partial answer, and conjectures that the axioms are always independent in case implication is a primitive symbol.

The author also establishes the following theorem, which is not very closely related to the main topic of the paper: a set Γ of formulas of the sentential calculus is simultaneously satisfiable if and only if it is not the case that every formula is derivable from Γ.

J. C. C. McKinsey.

Henkin, Leon. The completeness of the first-order functional calculus. J. Symbolic Logic **14**, 159–166 (1949).

A new proof is given of the completeness of the first-order functional calculus. The essential idea of the proof consists in showing that, if A is any consistent set of closed well-formed formulas, then, by increasing the number of individual constants, we can obtain a set Γ of closed well-formulas which, in addition to being consistent, is complete in the sense of Post, and satisfies the following condition: if a formula of the form $(Ex)A$ is in Γ, then Γ also contains a formula A' which is obtained from A by replacing x by some individual constant. It is pointed out that the proof does not depend on the assumption that the number of symbols in the first-order functional calculus is countable, which indicates that this proof must have some applications in abstract algebra. [In formula C_i on page 166, the second x_2 should be replaced by x_3.]

J. C. C. McKinsey.

Hoo, Tzu-Hua. *m*-valued sub-system of $(m+n)$-valued propositional calculus. J. Symbolic Logic **14**, 177–181 (1949).

It is shown that, if $m>n$, one can always define, within an *m*-valued propositional calculus, operations which will satisfy just those axioms satisfied by the primitive operations of an *n*-valued propositional calculus.

J. C. C. McKinsey (Santa Monica, Calif.).

Lalan, Victor. Équations fonctionnelles dans un anneau booléien. C. R. Acad. Sci. Paris **230**, 603–605 (1950).

It is noted that Sheffer's stroke-symbol in Boolean algebra can be interpreted to mean either $p' \cap q'$ or $p' \cup q'$, and asserted that, in the propositional calculus, one should postulate $p|(p|p)$ to distinguish which one is meant. The ideas are closely related to those of the author's paper in the same C. R. **223**, 1086–1087 (1946) [these Rev. **8**, 307].

G. Birkhoff (Cambridge, Mass.).

Trahtenbrot, B. A. The impossibility of an algorithm for the decision problem for finite domains. Doklady Akad. Nauk SSSR (N.S.) **70**, 569–572 (1950). (Russian)

The author asserts that there does not exist any general recursive algorithm for deciding whether a formula of the restricted functional calculus is true in every finite domain. This is related to Church's result [Amer. J. Math. **58**, 345–363 (1936); J. Symbolic Logic **1**, 40–41 (1936)] that no such algorithm exists for deciding whether a formula is derivable (and hence true in every domain, finite or infinite). An outline of the proof is given, but not the details. An application to the theory of finite sets is mentioned.

H. B. Curry (State College, Pa.).

Peremans, W. A remark on intuitionistic logic. Math. Centrum Amsterdam. Rapport **ZW-1949-016**, 4 pp. (1949). (Dutch)

In intuitionistic as well as in classical logic the formula $(x)f(x)\rightarrow(Ex)f(x)$ expresses that the system is only valid if the class of individuals contains at least one element. In order to make logic applicable to classes of which no element is known, the author proposes to bind the free variables in any formula by universal quantifiers ranging over the whole formula. This necessitates obvious changes in the rules of derivation. In particular, the separation rule deriving $(x)B(x)$ from $(x)A(x)$ and $(x)(A(x)\rightarrow B(x))$ is only valid if $B(x)$ really depends on x.

A. Heyting (Amsterdam).

Fraenkel, Abraham Adolf. Problems and methods in modern mathematics. Scripta Math. **15**, 169–182 (1949).

Preceding installments appeared in Scripta Math. **6**, 69–79 (1939); **7**, 9–20 (1940); **9**, 5–18, 81–84, 162–168, 245–255 (1943); these Rev. **2**, 210; **5**, 86, 198; **6**, 32. A concluding part is to appear later.

Bourbaki, Nicholas. The architecture of mathematics. Amer. Math. Monthly **57**, 221–232 (1950).

Translation of a chapter in the book Les grands courants de la pensée mathématique [Cahiers du Sud, 1948; these Rev. **10**, 230].

ALGEBRA

David, F. N., and Kendall, M. G. Tables of symmetric functions. I. Biometrika **36**, 431–449 (1949).

The tables presented here give the coefficients in the linear representation of the monomial symmetric functions in terms of products of sums of like powers and inversely the coefficients in the expression of products of sums of powers as linear combinations of monomial symmetric functions. The weights of the symmetric functions are $2, 3, \cdots, 12$. To avoid fractions, each monomial symmetric function is multiplied by an appropriate product of factorials. For example, the function $\sum\alpha^4\beta^2\gamma^2\delta\epsilon\zeta$ is multiplied by $1!\,2!\,3!$. The application of the tables to problems in "k-statistics" is discussed and illustrated. [For similar tables see also Ziaud-Din, Proc. Nat. Acad. Sci. India. Sect. A. **10**, 53–60 (1940); Kerawala, ibid. **11**, 51–55 (1941); Kerawala and Hanafi, ibid., 56–63 (1941); **12**, 81–96 (1942); these Rev. **8**, 191.]

D. H. Lehmer (Berkeley, Calif.).

Dickinson, David. On sums involving binomial coefficients. Amer. Math. Monthly **57**, 82–86 (1950).

The sums in question $\sum_k\binom{n+ak}{b+ck}$ are for integral values of all variables, with $n+ak$, $b+ck$ both nonnegative, and to keep the sums finite $c>a>0$. The evaluations are in terms of the quantity $m=nc-ab$, called the determinant of the sum; that is, the sum above is taken as S_m. This requires distinction of two cases according as a and c are or are not relatively prime. In the first of these, if α_r, $r=0$ to $c-1$, are the (distinct) roots of the equation $x^c-x^a-1=0$, then $S_m=\sum\alpha_r^m(c-a\alpha_r^{a-c})^{-1}$. In the second case, an expression, which with the necessary definitions is too long to quote, is given. It may be noted that in the old literature [cf. Netto, Lehrbuch der Kombinatorik, Teubner, Leipzig, 1901, p. 247] the sum is taken as a function of n with a, b, c fixed, say $u(n; a, b, c)$, which makes the distinction of relative primality of a, c unnecessary. Indeed, by a ready generalization, $U(a, b, c)=\sum t^n u(n; a, b, c)=t^b(1-t)^{c-1-b}[(1-t)^c-t^{c-a}]^{-1}$.

J. Riordan (New York, N. Y.).

Mignosi, Giuseppe. Estensione ai corpi finiti di una formula di Radòs. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 216–219 (1950).

An identity of G. Radòs [Rend. Circ. Mat. Palermo **46**, 308–314 (1922)] concerning congruences mod p (p a prime) is generalized by introducing a finite field of $N=p^m$ elements. In this field an equation $f(x)\equiv x^n+a_1x^{n-1}+\cdots+a_{n-1}x=q$ is considered when $n\leqq N-2$. The values q_i of q for which $f(x)=q_i$ has n distinct (nonzero) roots are called nth numbers with respect to $f(x)$. A polynomial $A(q)$ which satisfies $f(x)=q$ for all nth numbers with respect to $f(x)$ and the factorization of the polynomial $f(A(x))-x$ are studied.

O. Todd-Taussky (Washington, D. C.).

Mignosi, Giuseppe. Ancora sopra una estensione ai corpi finiti di una formula di Radòs. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 284–289 (1950).

The results of the paper reviewed above are applied to the study of the special case when $f(x)\equiv x^n$.

O. Todd-Taussky (Washington, D. C.).

MacDuffee, C. C. Some applications of matrices in the theory of equations. Amer. Math. Monthly **57**, 154–161 (1950).

The polynomial $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$ has associated with it a companion matrix of order n got by bordering the identity of order $n-1$ above by the row $(0, \cdots, 0, -a_n)$ and on the right by the column $(-a_n, \cdots, -a_1)$. This paper shows how the companion matrix can be employed to study the complex roots of polynomials and to compute polynomials symmetric in those roots. The following apparently new theorem is proved: if $f(x)$, $g(x)$ are polynomials with complex coefficients and A is the companion matrix of f, the number of complex roots which f, g have in common is the nullity of $g(A)$. A corollary is: the number of distinct roots of $f(x)$ is the rank of $f'(A)$.

J. M. Thomas.

Souriau, J.-M. Le calcul spinoriel et ses applications. Recherche Aéronautique **1950**, no. 14, 3–8 (1950).

In this paper index tensors under the linear group which are antisymmetric in all their indices are called spinors. The paper contains some well-known results in matrix algebra having to do with the solutions of systems of linear equations.

A. H. Taub (Urbana, Ill.).

The theorem described in the review as "apparently new" is equivalent to an earlier result of W.V. Parker, [Amer. Math. Monthly 42, 164–166 (1935)] and M.M. Flood [ibid. 43, 562–563 (1936)].

Cherubino, Salvatore. Forma quasi-canonica delle matrici. Ann. Scuola Norm. Super. Pisa (3) **2** (1948), 151–166 (1950).

A "quasi-canonical" form for general matrices is discussed which is obtained by transformation by unitary matrices. It coincides with the usual canonical form in the case of real symmetric, Hermitian and unitary matrices. Normal matrices are not discussed nor Schur's theorem that every matrix can be transformed to triangle form by a unitary matrix. The paper contains also various observations connected with Hadamard's theorem on the maximum value of a determinant. *O. Todd-Taussky* (Washington, D. C.).

Abstract Algebra

Dilworth, R. P. The structure of relatively complemented lattices. Ann. of Math. (2) **51**, 348–359 (1950).

First is a brief account, with some new results, of the general theory of lattice congruences. Thence known results on the structure of modular lattices are extended to relatively complemented lattices. Definitions: A quotient a/b is weakly projective into c/d if there exists a sequence of quotients $a/b=x_1/y_1, \cdots, x_n/y_n=c/d$ such that x_i/y_i is contained in a transpose of x_{i+1}/y_{i+1}; $\Theta(L)$ is the set of all congruence relations over the lattice L; the center $\Gamma(L)$ of L is the set of those elements of $\Theta(L)$ which permute with every element of $\Theta(L)$; if L is the direct sum of two components then the natural mapping into one component is a decomposition congruence relation. If a is an element of a finite dimensional lattice, let I_a denote the join of all elements covering a; dually for 0_a. Then the lattice is locally relatively complemented if I_a/a and $a/0_a$ are relatively complemented for each a.

The following theorems are proved. A lattice L is a direct union of a finite number of simple lattices if and only if $\Gamma(L)=\Theta(L)$ and $\Theta(L)$ is a finite Boolean algebra. The following statements are equivalent, regarding a congruence relation θ on a relatively complemented lattice L: (1) there exists a maximal element $m \in L$ such that $m \equiv 0\ (\theta)$; (2) there exists an element $a \in L$ such that $x \equiv y\ (\theta)$ if and only if $x \cap a = y \cap a$; (3) θ is a decomposition congruence relation. Every two congruence relations on a relatively complemented lattice permute. If a relatively complemented lattice satisfies a chain condition, then $\Theta(L)$ satisfies the same chain condition. A relatively complemented lattice satisfying the ascending chain condition is a direct union of a finite number of simple relatively complemented lattices. Let L be a relatively complemented lattice satisfying a chain condition; then L is simple if and only if every two prime quotients of L are projective. Let L be a locally relatively complemented lattice of finite dimensions; then a prime quotient is weakly projective into a quotient a/b if and only if it is projective to a prime quotient contained in a/b. Every locally relatively complemented lattice of finite dimensions is a subdirect union of a finite number of simple locally relatively complemented lattices.

P. M. Whitman (Silver Spring, Md.).

Michiura, Tadashi. Lattice ordered rings and ordered characterizations of integers. J. Osaka Inst. Sci. Tech. Part I. **1**, 29–31 (1949).

Postulates are given for the domain of integers, regarded as an ordered ring. [The condition that $x>0$ and $xy \geqq 0$ imply $y \geqq 0$, included by the author as a postulate defining an l-ring, seems inappropriate in this wider context, as it is not fulfilled by functions.] Postulates for an l-ring in terms of the unary operation a^* (interpreted as $a \cup 0$) are also given. *G. Birkhoff* (Cambridge, Mass.).

Gröbner, W. Über die Syzygientheorie der Polynomideale. Monatsh. Math. **53**, 1–16 (1949).

Let $R=K[x_0, \cdots, x_n]$ be a polynomial ring over a field K, and let U_1 be a matrix with elements in R and with all its subdeterminants homogeneous. The set of all 1-column matrices V such that $U_1V=0$ is an R-module with finite basis; the matrix U_2 whose columns are the elements of such a basis is called a syzygetic matrix of U_1. In general, let U_{i+1} be a syzygetic matrix of U_i. It is proved that there exists an integer $k \leqq n+1$ such that $U_{k+1}=0$. In case U_1 is 1-rowed this reduces to Hilbert's theorem on the syzygetic chain of a homogeneous ideal A (the elements of U_1 constituting a basis for A). If A has a primary component of rank r (i.e., projective dimension $n-r$) then it is proved that $k \geqq r$; if A is of the principal class and rank r, then $k=r$. In general, A is called perfect if k equals its rank, and it is shown that this definition of perfect ideal is equivalent to Macaulay's. The preceding theory is applied to space curves given by perfect ideals. *I. S. Cohen.*

Dubreil, Paul. Sur quelques problèmes concernant les variétés algébriques et la théorie des syzygies des idéaux de polynomes. C. R. Acad. Sci. Paris **229**, 11–12 (1949).

The author points out that certain results of Gröbner [see the preceding review] have been obtained before. He also observes that a question left unanswered by Gröbner had already previously been settled in the negative.

I. S. Cohen (Cambridge, Mass.).

Levitzki, Jakob. On multiplicative systems. Compositio Math. **8**, 76–80 (1950).

The date of receipt of this paper is 1939. Answering a problem raised by Köthe, the author shows that the maximal condition implies that nil ideals in a ring are nilpotent. Improved results were given by the author in a later paper [Amer. J. Math. **67**, 437–442 (1945); these Rev. **7**, 2].

I. Kaplansky (Chicago, Ill.).

Borofsky, S. Characterization of a field by a single operation. Amer. J. Math. **71**, 92–104 (1949).

The concept of field is characterized by six simple postulates as a system $(F, \circ)$ with single composition, $\circ$, which permits the realization (i) $a \circ b = a(b-1)$, as well as (ii) a simple linear family of related realizations. Subfields are characterized in terms of $\circ$ much as subgroups in a group.

A. L. Foster (Berkeley, Calif.).

Wang, Shianghaw. On Grunwald's theorem. Ann. of Math. (2) **51**, 471–484 (1950).

The problem of Grunwald was essentially as follows. Let F be finite algebraic over the rationals, l a rational prime, and S a finite set of prime divisors in F. For each $\mathfrak{p} \in S$ choose a field $K^{\mathfrak{p}}$ which is cyclic, of degree a power of l, over the $\mathfrak{p}$-adic completion $F_{\mathfrak{p}}$ of F and let $l^{\mathfrak{p}}$ be the maximum $\deg(K^{\mathfrak{p}}/F_{\mathfrak{p}})$. When does there exist an extension K, cyclic over F of degree $l^{\mathfrak{p}}$, which has as its $\mathfrak{P}$-adic extensions exactly the fields $K^{\mathfrak{p}}$, for all $\mathfrak{P} \mid \mathfrak{p} \in S$, and for which the local Galois groups are in prescribed position in the Galois group of K/F?

The author has already shown [Ann. of Math. (2) 49, 1008–1009 (1948); these Rev. 10, 231] that Grunwald's theorem (that such a K always exists) is false. He proves here the correct theorem which replaces it. If $l \neq 2$, such a K always exists. If $l=2$, K always exists unless (a) $F(\zeta)$ is noncyclic over F, where ζ is a primitive 2^sth root of 1, (b) the set of primes $\mathfrak{q}$ (which must be divisors of 2, and are called the oddly-even primes) such that degree $(F_{\mathfrak{q}}(\zeta):F_{\mathfrak{q}})$ = degree $(F(\zeta):F)$ is nonempty, and (c) all the oddly-even primes are in S. In this case, if we consider F as subfield of the complex field, there is an integer $t<s$ such that $\sec(2\pi/2^t) \in F$ and $\sec(2\pi/2^{t+1}) \notin F$. Let $\alpha=(\sec(2\pi/2^{t+1}))^{2^t}$. This α is in F, and has the property of being a 2^tth power in $F_{\mathfrak{p}}$ for all $\mathfrak{p}$ not oddly-even, and a 2^{t-1}th power in $F_{\mathfrak{q}}$ for all oddly-even $\mathfrak{q}$. The product formula for the norm residue symbol easily gives, as necessary condition for the existence of a field K, that the fields $K^{\mathfrak{q}}$ must be so chosen that α is a local norm in $K^{\mathfrak{q}}/F_{\mathfrak{q}}$ for an even number of oddly-even primes $\mathfrak{q}$. It is shown that this condition is also sufficient.

The author actually proves more than this: the complete theorem determines a necessary and sufficient condition that there exist a field K which, besides having the local components $K^{\mathfrak{p}}$ at each $\mathfrak{p} \in S$, shall contain a preassigned cyclic extension C of F and shall have certain local properties with respect to a given cyclic extension $\bar{C}$ of F.

The proof that every rational division algebra is cyclic over its center demands only that there exist a cyclic K with large enough local degree at each $\mathfrak{p} \in S$. This paper shows that such a K always exists, and thus completes the first correct proof of this celebrated theorem for algebras of even degree. A proof of the norm theorem of Hasse-Schilling-Maass is also given.

Let A be a central simple algebra over an algebraic number field F. Then $\alpha \in F$ is a norm in A if and only if it is positive at all infinite primes where A is ramified. As far as the reviewer knows, this is the first correct proof of this theorem too. Proofs are complete and nonanalytic. Misprint: The second exponent $t+1$, in the third line of lemma 1, should be s. *G. Whaples* (Bloomington, Ind.).

Hochschild, G. Local class field theory. Ann. of Math. (2) 51, 331–347 (1950).

The theorems of local class field theory are proved here by a method which modifies in two ways Chevalley's [J. Reine Angew. Math. 169, 140–157 (1933)] derivation by means of simple algebras. First, the theory of noncommutative algebras is eliminated and is replaced by the Eilenberg-MacLane [Ann. of Math. (2) 48, 51–78, 326–341 (1947); these Rev. 8, 367; 9, 7] cohomology theory for groups. If L is a field and $\mathfrak{L}$ a finite group of its automorphisms, the 2-cocycles are the factor sets which in the theory of algebras generate crossed products of L and $\mathfrak{L}$, and the bounding 2-cocycles are the factor sets which generate algebras ~ 1. The second change consists in replacing the norm residue symbol by the Akizuki-Nakayama [Akizuki, Math. Ann. 112, 566–571 (1936); Nakayama, ibid., 85–91 (1935); Jap. J. Math. 18, 877–885 (1943); these Rev. 7, 363] formula which assigns to each factor set a mapping of the Galois group into the norm class group. The most important steps now consist in working out a transfer rule between the 2-cohomology groups of different extension fields (which replaces the theory of splitting fields) and defining in terms of the cohomology groups a mapping which is equivalent to the norm residue symbol. The fact that an algebraic meaning for the 3-cohomology group has already been found [Eilenberg and MacLane, Trans. Amer. Math. Soc. 64, 1–20 (1948); these Rev. 10, 5] shows the fruitfulness of the new method. *G. Whaples* (Bloomington, Ind.).

Raffin, Raymond. Anneaux à puissances commutatives et anneaux flexibles. C. R. Acad. Sci. Paris 230, 804–806 (1950).

Nonassociative rings having certain properties of commutativity or associativity for powers of any element are considered. It is shown that, if A is a flexible ring (that is, $(xy)x=x(yx)$ holds) and if there is no element of characteristic two, then any n elements of A which commute with each other generate a commutative subring of A.

R. D. Schafer (Philadelphia, Pa.).

Raffin, Raymond. Sur certaines propriétés de commutation dans les anneaux monogènes. C. R. Acad. Sci. Paris 230, 904–906 (1950).

Commutators of powers of an element in a nonassociative ring are considered. Examples show that some of the properties of commutativity of powers defined in the paper reviewed above are actually inequivalent.

R. D. Schafer (Philadelphia, Pa.).

Rees, D. Linear systems of algebras. Ann. of Math. (2) 51, 123–160 (1950).

Let $\mathfrak{A}_1, \cdots, \mathfrak{A}_r$ be (nonassociative) algebras over a field $\mathfrak{F}$, each of which has the same vector space $\mathfrak{A}$ as underlying vector space. Let the product of two elements a, b of $\mathfrak{A}_i$ be written aM_ib. Then if $\alpha_1, \cdots, \alpha_r$ are elements of $\mathfrak{F}$, a new algebra with vector space $\mathfrak{A}$ can be defined with the product $aMb=\sum_i \alpha_i(aM_ib)$. The set of all algebras which may be defined in this way is called a linear system Σ of algebras. Simplicity, semi-simplicity, and irreducibility are defined for such a system, and conditions are obtained that the system have these properties. These conditions and certain structure theorems which are proved are analogous to those of R. H. Bruck for algebras [Trans. Amer. Math. Soc. 56, 141–199 (1944); these Rev. 6, 116].

If M_{ij}, $i=1, \cdots, s$; $j=1, \cdots, r$, are multiplications of Σ which span the space of all multiplications of the system, an algebra can be constructed whose elements $A, B, \cdots$ are $r \times s$ matrices with elements in $\mathfrak{A}$ and in which multiplication is defined by the formal matrix product $A \cdot B=A(M_{ij})B$. This algebra is called a matrix expansion of Σ. The principal result following this definition is that an algebra which is both right and left simple is isomorphic to a matrix expansion of an irreducible system of algebras.

F. Kiokemeister (South Hadley, Mass.).

Barsotti, Iacopo. Struttura delle algebre semplici di genere basso su corpi di funzioni algebriche di una variabile. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 8, 134–167 (1949).

Based upon the generalized Riemann-Roch theorem this paper treats a novel classification of central simple algebras S of finite degree n over a field of algebraic functions K of genus g with a perfect coefficient field k. Basic for the considerations of the author is Witt's formula for the genus of S, which is written in the form $G=ng-n+1+d$, where d is the degree of the (total) different of S/K, which is measured by the degree of the local reduced norms of the components. This formula implies that each algebra of genus 0 is a division algebra and that the matrix algebras $k(x)_n$ are the only simple algebras of degree n (over their centers) and genus $1-n$. In order to obtain detailed results

for $G=0, 1, 2$ the field k is restricted to be a finite field so that the theory of local invariants becomes applicable. Thus $d=\frac{1}{2}\sum_p \nu_p[n-(n, \epsilon_p\nu_p)]$, where ν_p is the degree of the prime divisor p of K and ϵ_p the p-adic order of the factor set α in the normalized representation $S=(Z, \sigma, \alpha)$ with an unramified cyclic field Z/K. It follows immediately that the algebras of negative genus G are direct products of division algebras of genus 0 and the matrix algebra of genus $1-|G|$. If $G=0$ then there are only the algebras (Z, σ, α) with $\alpha=((x-a)/(x-b))^r$, $(x-a)^r$, $(r, [Z:k(x)])=1$, $a\neq b$ in k. For $G=1$ a complete list can be given since the degree n proves to be limited to be one of the numbers 1, 2, 3, 4, 6, where $n=1$ determines the elliptic fields over k. Furthermore, among other detailed results, the existence of a function $\lambda(G)$ is shown such that whenever $n>\lambda(G)$ there is no division algebra of degree n and genus G. Finally the author observes that algebras with cyclic unramified splitting fields Z over a rational field $k(x)$, for general k, can be treated as in the special case of finite fields k since the explicit formula for the genus G holds. In this slightly more general case the classical results on exponent and index hold if $G=0$.

O. F. G. Schilling (Chicago, Ill.).

Amitsur, Shimshon. La représentation d'algèbres centrales simples. C. R. Acad. Sci. Paris **230**, 902–904 (1950).

Let A be a central simple algebra over a field C of characteristic 0. The main theorem asserts that A can be represented as the endomorphism ring of a suitable differential transformation in the sense of Jacobson [Ann. of Math. (2) **38**, 484–507 (1937)]. To prove this, the author takes a transcendental splitting field K of A, and a differentiation in K with constant field C. The mapping $k\to\sum k^{(n)}t^n/n!$ is an isomorphism of K into the power series field $K[t]$ and induces a corresponding isomorphism of the matrix ring K_n. Since $A\times K\cong K_n$, A is isomorphic to a subalgebra of K_n, and this gives rise to two representations of A over $K[t]$. Their equivalence leads to the desired differential transformation. It is conversely true that every such algebra of endomorphisms will have K as splitting field.

I. Kaplansky (Chicago, Ill.).

Amitsur, Shimshon. Construction d'algèbres centrales simples sur des corps de caractéristique zéro. C. R. Acad. Sci. Paris **230**, 1026–1028 (1950).

The author deduces from the foregoing result [cf. the preceding review] that A can also be exhibited as the invariant ring of a suitable differential polynomial.

I. Kaplansky (Chicago, Ill.).

Kreĭn, M. A principle of duality for bicompact groups and quadratic block algebras. Doklady Akad. Nauk SSSR (N.S.) **69**, 725–728 (1949). (Russian)

Soit $\mathfrak{A}$ une algèbre commutative sur le corps complexe munie d'un élément unité e et d'une involution $a\to\bar{a}$; l'auteur dit que $\mathfrak{A}$ est une "algèbre de blocs" si $\mathfrak{A}$ possède une base (au sens algébrique) que l'on peut découper en une suite de parties deux à deux disjointes ("blocs"), de telle sorte que les conditions suivantes soient réalisées: (a) le nombre d'éléments de chaque bloc est un carré parfait; on peut donc ranger les éléments d'un bloc sous la forme d'une matrice carrée $(u_{ij}^{(\nu)})$ $(1\leq i, j\leq n_\nu$, où ν est un indice qui sert à repérer le "bloc" considéré); on notera U_ν cette matrice; (b) il existe un bloc qui est réduit à l'élément e de $\mathfrak{A}$; (c) si $U=(u_{ij})$ est un bloc, il en est de même de $\bar{U}=(\bar{u}_{ij})$, et en désignant par U^* la matrice transposée de U (et non de U comme il est dit à la suite d'une erreur d'impression) on a la relation $UU^*=(\delta_{ij}\cdot e)$, où δ_{ij} est le symbole de Kronecker; (d) si U et V sont deux blocs, la matrice $U\otimes V$ (produit tensoriel) se réduit, par une transformation unitaire d'ordre convenable, à une matrice de la forme $W_1\oplus\cdots\oplus W_n$, où les W_i sont des blocs (ici encore, il y a une faute d'impression; il faut remplacer "$W_i\varepsilon\mathfrak{A}$" par "$W_i\varepsilon\mathfrak{U}$"); (e) avec les notations de (d), si $V=\bar{U}$ il existe un bloc W_i et un seul qui se réduit à e; si $V\neq\bar{U}$, tous les W_i sont $\neq e$.

Si G est un groupe compact, et si $\mathfrak{A}$ est l'ensemble des combinaisons linéaires de coefficients des diverses représentations irréductibles de G, il est visible qu'on peut considérer $\mathfrak{A}$ comme une algèbre de blocs (la multiplication étant le produit ordinaire de deux fonctions). Le théorème démontré ici est que, réciproquement, toute algèbre de blocs s'obtient à partir d'un groupe compact. Le groupe G associé à une telle algèbre $\mathfrak{A}$ peut s'obtenir de deux façons: (a) les éléments de G sont les "représentations" de $\mathfrak{A}$ (on associe à chaque bloc U une matrice unitaire complexe $f(U)$ de même ordre, de telle sorte que les relations algébriques entre blocs se traduisent en relations analogues sur les $f(U)$; la multiplication dans G s'obtient alors en appelant produit de deux représentations f et g la représentation $U\to f(U)g(U)$); (b) N désignant l'ensemble des indices ν des divers blocs, on considère les fonctions $\nu\to A_\nu$, définies sur N et donc la valeur en ν est une matrice complexe A_ν, de même ordre que U_ν, et pour lesquelles on a $\sum_{\nu\varepsilon N}\mathrm{Tr}\,(A_\nu^*A_\nu)^{\frac{1}{2}}<+\infty$; ces fonctions forment un espace de Banach $\mathfrak{R}$ de façon évidente, et l'on peut identifier $\mathfrak{A}$ a un sous-espace partout dense de $\mathfrak{R}$ en associant à chaque $x\varepsilon\mathfrak{A}$ la fonction A_ν telle que l'on ait $x=\sum_{\nu\varepsilon N}\mathrm{Tr}\,(A_\nu U_\nu)$; ceci fait, on constate que la structure d'algèbre de $\mathfrak{A}$ se prolonge par continuité à $\mathfrak{R}$, d'où une algèbre normée complète commutative avec unité: G s'identifie alors à l'espace des idéaux maximaux de $\mathfrak{R}$, et il reste à prouver que $\mathfrak{A}$ est identique à l'algèbre des représentations de G, ce qui est facile. Bien entendu, on obtient ainsi une démonstration de plus du théorème bien connu de Tannaka. On peut d'autre part se demander s'il ne serait pas possible de passer du cas discret au cas continu: i.e., par dualité, des groupes compacts aux autres.

R. Godement (Nancy).

Harish-Chandra. On the radical of a Lie algebra. Proc. Amer. Math. Soc. **1**, 14–17 (1950).

Soient L une algèbre de Lie sur un corps de caractéristique nulle, Γ son radical, N l'ensemble des $X\varepsilon\Gamma$ tels que l'opérateur $Y\to[X, Y]$ soit nilpotent; N est l'unique idéal nilpotent maximal de L. D'autre part, conformément aux notations de C. Chevalley, soit ad X l'opérateur $Y\to[X, Y]$; posons $B(X, Y)=\mathrm{sp}(\mathrm{ad}\,X\cdot\mathrm{ad}\,Y)$, forme fondamentale de L, et $T(X, Y, Z)=\mathrm{sp}(\mathrm{ad}\,[X, Y]\cdot\mathrm{ad}\,Z)$; cette dernière forme trilinéaire est alternée, invariante par toute dérivation de L; l'auteur démontre d'abord que le radical Γ est l'ensemble des X tels que $T(X, Y, Z)=0$ quels que soient Y, Z; d'où résulte que Γ et N sont invariants par toute dérivation de L. Le second résultat est le suivant. Pour un $X\varepsilon N$, ad X est nilpotent, donc on peut former l'opérateur exp (ad X); les opérateurs ainsi obtenus engendrent un groupe d'automorphismes G de L; ceci étant, si l'on a une décomposition de Levi $L=S+\Gamma$ (S semi-simple) et si M est une sous algèbre semi-simple quelconque de L, alors on peut amener M dans S par un élément de G. Il en résulte une forme un peu plus forte du théorème de Malcev suivant lequel deux sous-algèbres semi-simples maximales sont conjuguées.

R. Godement (Nancy).

Harish-Chandra. Lie algebras and the Tannaka duality theorem. Ann. of Math. (2) **51**, 299–330 (1950).

Le but de cet article est d'étendre aux algèbres de Lie semi-simples, et aux groupes de Lie semi-simple non nécessairement compacts, les résultats de Tannaka sur la dualité des groupes compacts non abéliens. Le § 1 donne les définitions suivantes. Soient $\mathfrak{L}$ une algèbre de Lie sur un corps k, et $\mathfrak{F}$ l'ensemble de toutes ses représentations linéaires de dimension finie dans des espaces vectoriels sur un corps donné $K \supset k$; on peut définir dans $\mathfrak{F}$ les opérations suivantes: (a) somme directe; (b) produit tensoriel; (c) passage d'une représentation $X \to \rho(X)$ à une représentation semblable $\gamma \cdot \rho \cdot \gamma^{-1}$, où γ est une matrice inversible donnée dans l'espace de la représentation ρ; (d) passage de la représentation ρ à sa transposée ρ^*. Une partie $\mathfrak{R}$ de $\mathfrak{F}$ est un anneau si elle est invariante par les opérations (a) et (b), et si en outre toute représentation semblable à un facteur direct d'une représentation appartenant à $\mathfrak{R}$ est aussi dans $\mathfrak{R}$. Enfin, on appelle représentation d'un tel anneau $\mathfrak{R}$ toute application $\rho \to \xi(\rho)$ qui, à chaque élément $\rho \in \mathfrak{R}$, associe une matrice $\xi(\rho)$ dans l'espace de la représentation ρ, et ceci de telle sorte que cette application soit compatible avec les opérations (a), (b), (c). Les représentations de $\mathfrak{R}$ forment une algèbre de Lie $\tilde{\mathfrak{L}}(\mathfrak{R})$ sur K, le crochet étant défini par $[\xi_1, \xi_2](\rho) = [\xi_1(\rho), \xi_2(\rho)]$.

Dans le § 2, l'auteur démontre essentiellement que tout anneau $\mathfrak{R}$ de représentations, invariant par $\rho \to \rho^*$, possède un "générateur," i.e., qu'il existe une représentation ρ de $\mathfrak{L}$ telle que $\mathfrak{R}$ soit le plus petit anneau contenant ρ; ceci, bien entendu, dans l'hypothèse où $\mathfrak{L}$ est semi-simple. La démonstration est assez longue et utilise de façon essentielle l'opérateur de Casimir. Le résultat donne (en particulier) une démonstration générale du fait que, si ρ est une représentation bien choisie de $\mathfrak{L}$, toutes les autres s'en déduisent par des opérations standard; on sait que ce résultat avait été vérifié par É. Cartan à l'aide de la classification.

Le § 3 démontre le théorème de dualité, à savoir: soient $\mathfrak{L}$ une algèbre de Lie semi-simple sur un corps k de caractéristique 0, K une extension algébriquement fermée de k, $\mathfrak{R}$ un anneau de K-représentations de $\mathfrak{L}$, invariant par l'opération *; tout élément $X \in \mathfrak{L}$ définit de façon évidente une représentation de $\mathfrak{R}$; les représentations ainsi obtenues forment dans l'algèbre de Lie $\tilde{\mathfrak{L}}(\mathfrak{R})$ un ensemble $\mathfrak{L}(\mathfrak{R})$ qui est évidemment une algèbre de Lie sur k; ceci étant, $\tilde{\mathfrak{L}}(\mathfrak{R})$ n'est autre que l'algèbre déduite de $\mathfrak{L}(\mathfrak{R})$ par passage de k à K.

Le § 4 concerne les groupes de Lie semi-simples réels. Soit G un tel groupe; dans l'ensemble $\mathfrak{F}$ des représentations linéaires de G par des matrices complexes, on peut définir cinq opérations, à savoir somme directe, produit tensoriel, passage à une représentation équivalente, ou à la représentation transposée, ou à la représentation imaginaire conjuguée; on définit les sous-anneaux de $\mathfrak{F}$ comme dans le cas des algèbres de Lie, de même que les représentations d'un tel sous-anneau $\mathfrak{R}$: celles-ci forment, comme dans le cas de l'anneau des représentations d'un groupe compact, un groupe $\tilde{G}(\mathfrak{R})$. L'auteur montre tout d'abord qu'on peut munir de façon naturelle $\tilde{G}(\mathfrak{R})$ d'une structure de groupe de Lie réel; puis que l'algèbre de Lie de $\tilde{G}(\mathfrak{R})$ s'identifie canoniquement à $\tilde{\mathfrak{L}}(\mathfrak{R}')$, où $\mathfrak{R}'$ désigne l'ensemble des représentations de l'algèbre de Lie $\mathfrak{L}$ de G définies par les éléments de $\mathfrak{R}$; mais puisque $\tilde{\mathfrak{L}}(\mathfrak{R}')$ est une algèbre de Lie complexe, ceci permet d'introduire sur $\tilde{G}(\mathfrak{R})$ une structure complexe. Le résultat fondamental est alors le suivant: soit $\mathfrak{R}$ un anneau invariant par *; soit $x \to \rho(x)$ une représentation de G telle que $\mathfrak{R}$ soit "engendré" par ρ (cf. § 2); soit H le groupe linéaire image de G par ρ, et soit $\bar{H}$ le groupe linéaire déduit de H par passage du réel au complexe (si H est dans l'espace de dimension n sur le corps complexe C, on peut réaliser son algèbre de Lie par des matrices dans cet espace; les combinaisons linéaires complexes de ces matrices forment alors une algèbre, qui engendre un sous-groupe de $GL(n, C)$: ce sous-groupe est par définition $\bar{H}$); alors $\tilde{G}(\mathfrak{R})$ est connexe, et isomorphe à $\bar{H}$. En particulier, soient G un groupe semi-simple connexe, $\tilde{G}_R$ le sous-groupe de $\tilde{G}(\mathfrak{F})$ ($\mathfrak{F}$: anneau de toutes les représentations de G) formé des représentations "réelles" de $\mathfrak{F}$; alors la composante connexe de l'unité de $\tilde{G}_R$ est isomorphe à G/N, où N est le sous-groupe de G sur lequel toutes les représentations de G se réduisent à l'identité. *R. Godement.*

Matushima, Yozô. On the faithful representations of Lie groups. J. Math. Soc. Japan **1**, 254–261 (1949).

This paper contains a simplification of É. Cartan's proof of the fact that every Lie algebra L over the field of complex numbers has a faithful representation. The algebra L is first imbedded in a Lie algebra L_1 which may be written in the form $S+A+N$, where S is semi-simple, A Abelian, $[A, S]=0$ and N is a nilpotent ideal. Set then $T=S+A$, and let $\mathfrak{L}$, $\mathfrak{N}$ and $\mathfrak{T}$ be local Lie groups with the Lie algebras L_1, N and T, respectively; it may be assumed that $\mathfrak{L}=\mathfrak{N}\mathfrak{T}$ and $\mathfrak{N} \cap \mathfrak{T} = \{e\}$ (the neutral element). If a, x are in $\mathfrak{N}$, b, y in $\mathfrak{T}$, all sufficiently near e, $(ab)(xy) = F(ab; x)(by)$; for a given ab, the canonical coordinates of $F(ab; x)$ are polynomials in those of x. It follows by Cartan's lemma that the local group of transformations $x \to F(ab; x)$ has a faithful linear representation; this gives a representation of L_1 whose kernel is contained in T. Combining this representation with a faithful representation of L_1/N, one obtains a faithful representation of L_1. The method also yields a certain number of results on representations of Lie groups in the large; these results do not go substantially further than those of É. Cartan [J. Math. Pures Appl. (4) **17**, 1–12 (1938)] and A. Malcev [C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 87–89 (1943); these Rev. **7**, 115].

C. Chevalley (New York, N. Y.).

Dynkin, E. B. Some properties of the system of weights of a linear representation of a semisimple Lie group. Doklady Akad. Nauk SSSR (N.S.) **71**, 221–224 (1950). (Russian)

Let G be a semisimple Lie algebra, H a Cartan subalgebra, Π a fundamental system of roots, ϕ an irreducible representation of G, Λ a highest weight of ϕ. Any weight M is of the form $M = \Lambda - \alpha_1 - \alpha_2 - \cdots - \alpha_k$ ($\alpha_i \in \Pi$). Let s_k be the number of weights for which k α's occur, and $r+1$ the maximum value of k. The main theorem asserts that $s_k = s_{r-k}$, that s_i is nondecreasing from $i=1$ to $[r/2]+1$, and that $r = \sum_{\alpha \in \Pi} 2r_\alpha(\Lambda, \alpha)/(\alpha, \alpha)$, where r_α are certain integers depending only on G. The values of r_α are tabulated for each simple algebra (including the exceptional ones). In a preliminary theorem, the author considers a semisimple subalgebra G_1 of G, and a Cartan subalgebra $H_1 \subset H$. Then the system of weights of the induced representation of G_1 is obtained by orthogonal projection from H to H_1 (relative to the fundamental bilinear form). This is applied to the main problem, with G_1 a suitably selected three-dimensional simple subalgebra of G; the assertions of the theorem are deduced from known facts concerning representations of G_1. In a final theorem the author supposes further that ϕ is

self-contragredient in the sense of Malcev [Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] **8**, 143–174 (1944); these Rev. **6**, 146]. By applying a similar method, he deduces from the results of Malcev that ϕ is orthogonal if r is even, symplectic if r is odd.

I. Kaplansky (Chicago, Ill.).

Theory of Groups

Tartakovskiĭ, V. A. The sieve method in group theory. Mat. Sbornik N.S. **25(67)**, 3–50 (1949). (Russian)

Tartakovskiĭ, V. A. Application of the sieve method to the solution of the word problem for certain types of groups. Mat. Sbornik N.S. **25(67)**, 251–274 (1949). (Russian)

Tartakovskiĭ, V. A. Solution of the word problem for groups with a k-reduced basis for $k>6$. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 483–494 (1949). (Russian)

These papers contain a full account of the original method by which the author has attacked the word problem in groups. [Preliminary reports have appeared in Doklady Akad. Nauk SSSR (N.S.) **58**, 1605–1608, 1909–1910 (1947); these Rev. **9**, 321; **10**, 500.] No reference is made to earlier work in the field; nor indeed is such reference required. For the papers are completely self-contained and even the basic definitions of linear and cyclic words, their various equivalence relations and their simplest properties are developed ab ovo. [For a survey over these see, e.g., Magnus [Encyklop. Math. Wiss., 2d edition, Band I, Teil 1, Heft 4, Teil 1, 1939; these Rev. 1, 103].]

(1) Before we can indicate the types of groups for which the author has solved the word problem and the method by which these solutions are obtained we have to deal briefly with some known terminological and conceptual preliminaries. Throughout we consider a finite number of generators $S_1, S_2, \cdots, S_m$ to which formal orders (positive integers or ∞) are attached. For finite n the basic system of residue classes x with $x \not\equiv 0$, $-n/2 \leqq x < n/2$, is denoted by $V'(n)$. Two words $W_1(S)$ and $W_2(S)$ in the generators S_i are called (i) graphically equal, $W_1 \approx W_2$, if they are identical; (ii) formally equal, $W_1 \sim W_2$, if they can be made graphically equal by contracting powers of adjacent equal generators, or splitting a power of a generator into two, and by replacing the exponent of a power by a congruent one, modulo the order of its generator; (iii) freely equal, $W_1 = W_2$, if they can be made formally equal by omitting or inserting powers with zero exponent. Let now G be the group with generators $S_1, S_2, \cdots, S_m$ and the finite number of relations (1) $S_i^{n_i}=1$, where i runs through those indices for which the associated formal orders are finite; (2) $f_k(S_1, S_2, \cdots, S_m)=1$, $k=1, 2, \cdots, n$. (iv) The words W_1 and W_2 are equal in G, $W_1 = W_2$, if they can be made into freely equal words by omitting or inserting words of the form

$$L_1^{-1}(S) f_{k_1}^{\epsilon_1}(S) L_1(S) L_2^{-1}(S) f_{k_2}^{\epsilon_2}(S) L_2(S) \cdots L_r^{-1}(S) f_{k_r}^{\epsilon_r}(S) L_r(S), \quad \epsilon_\rho = \pm 1.$$

Here W^{-1} denotes the formal inverse of W; but if all exponents after inversion are replaced by the congruent ones in $V'(n)$ (n being the formal order of the generator concerned), the resulting word is written as $W^{\cong 1}$. (v) Cyclic words $|W(S)|$ are obtained by uniting into a class those words which differ by a cyclic permutation of their constituent powers. A cyclic word can be split after an arbitrary letter, possibly within a power, and straightened into a linear word. (If all generators of a cyclic word have finite formal orders then the number of formally unequal words obtainable by splitting is finite, otherwise infinite.) Conversely a linear word can be rolled into a cyclic word. A word is internally reduced if all adjacent powers belong to different generators (they are then called links) and their exponents lie in the appropriate $V'(n)$. To each word W there exists a uniquely determined reduced word $\bar{W}$ freely equal to W. The number of links in $\bar{W}$ is called the reduced length of W, $l(W)$. Two words W_1 and W_2 are said to be in mesh, if the last letter of W_1 is the same as the first letter of W_2. A linear word is externally reduced if the cyclic word resulting from it is reduced; if the first and last letter happen to be equal, then the sum of their exponents is not congruent to zero modulo the formal order of the generator concerned.

(2) A multiplication process, called composition, is now introduced for linear and cyclic words. In the case of two linear words W_1 and W_2 this can be described as follows: (i) the words are rolled into cyclic words $|W_1|$ and $|W_2|$; (ii) these are split at arbitrary places and straightened into linear words; (iii) they are then multiplied by juxtaposition and (iv) the result is rolled into a cycle which is then (v) split again at any place and straightened into a linear word. In a cyclic composition only the steps (ii)–(iv) are carried out. The composition is written as $W_1 * W_2$. For more than two factors composition is defined by recurrence and written with brackets. In this process a factor may be split into several components. The following theorems hold. (I) The total number of components in a composition of N factors is at most $2N-2$ (or $2N-1$ for linear compositions). (II) The process of composition is quasi-associative: every composition of three cyclic factors of the form $A*(B*C)$ is graphically equal to $(A*B)*C$ or $(A*C)*B$. A composition of the form $(\ldots((A_1*A_2)*A_3)*\ldots)*A_n$ is called monotonic and written without brackets. (III) Every composition of cyclic factors is graphically equal to a monotonic composition of the same factors where, moreover, any one of the given factors can be chosen as first factor. A word W with $W=1$ in G is called a Dyck word. Under ordinary multiplication every Dyck word is a product of factors each of which is a conjugate of the left hand side of one of the defining relations or its inverse. One of the main advantages of the composition process is that this explicit reference to the inner automorphisms of G can be avoided. (IV) Every Dyck word is freely equal to some composition of factors each of which is the left hand side of one of the defining relations or its graphical inverse.

If two reduced words W_1 and W_2 are in mesh, then their product W_1W_2 is reducible. If as many links as possible of the left factor are removed by reduction the resulting word is said to be completely reduced on the left. Similarly a complete reduction on the right is defined (the two may coincide). Let there be in a cyclic composition of N factors altogether κ components, $\kappa \leqq 2N-2$. By going round the word in a clockwise direction and systematically carrying out complete reductions on the left (or right) between successive components one arrives after a finite number of steps at an irreducible word. About the total number of reductions E which are applied between the components the author proves by a simple combinatorial argument: $E \leqq 2\kappa \leqq 4N-4$, a relation which is of great importance with the following. [Estimates for T in (7).]

(3) We are now in a position to describe the types of groups for which the author has solved the word problem.

Roughly speaking they are groups in which there is very little interdependence or superposition between the defining relations. Two ways are indicated of making this "very little" into a precise statement; but it is clear that the same idea could be expressed in other ways as well. Let the group be given as before. Let the reduced length of $f_\nu(S)$ be $l(f_\nu(S))=\lambda_\nu$ and let the relations be so numbered that $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$. It is shown that there is no loss of generality in assuming that the left hand sides of the relations are internally and externally reduced.

(I) Consider all pairs $|f_i^\epsilon(S)|$ and $|f_j^\eta(S)|$, $\epsilon, \eta = \pm 1$, $i, j = 1, 2, \cdots, n$. Split the cyclic words at arbitrary places subject only to the restriction that for $i=j$, $\epsilon=-\eta$ the places should not be corresponding, (i.e., such that the product of the split cycles would be freely equal to 1). Let $\lambda_{i,j}^{\epsilon\eta}$ be the length of the reducible part (the number of links which disappear from both factors) in the complete reduction on the left (or right) of the product of the split cycles; let $\mu_{i,j}^{\epsilon\eta}$ be the largest value of $\lambda_{i,j}^{\epsilon\eta}$ if all possible ways of splitting the two cycles are considered. Let

$$\delta = \max |\mu_{i,j}^{\epsilon\eta}/\lambda_i, \mu_{i,j}^{\epsilon\eta}/\lambda_j|$$

for $\epsilon, \eta = \pm 1$, $i, j = 1, 2, \cdots, n$. If $\delta < 1/8$ for the group G then the word problem can be solved in G.

(II) Consider a fixed left hand side $|f_p(S)|$ and multiply it by an arbitrary $|f_{q_1}^{\epsilon_1}(S)|$ after splitting the two cycles, again with the obvious restriction that for $q_1=p$, $\epsilon_1=-1$ the splitting points should not be corresponding. In the product carry out a partial or complete reduction. Operate on the remaining component of $|f_p(S)|$ similarly with an arbitrary $|f_{q_2}^{\epsilon_2}(S)|$ and carry out a further reduction. It may happen that after a finite number of such steps (multiplication and subsequent reduction) the original $|f_p(S)|$ has completely disappeared. Let K_p be the smallest number of steps in which this can be done and let

$$K = \min [K_1, K_2, \cdots, K_n].$$

If $K \geqq 9$ for the group G then the word problem can be solved in G. [In the third paper under review the author proves by a refinement of the combinatorial arguments that the method applies for $K=8$ and 7 as well, so that the word problem is soluble for $K>6$.]

(4) The method on which these solutions are based bears some resemblance to the sieve of Eratosthenes which explains the titles of the first two papers. An ordering is introduced into the set of all reduced words in the generators $S_1, \cdots, S_m$, first with respect to reduced length and for words of equal length anti-lexicographically. (Words which are not reduced are ordered with respect to their reduced forms.) In this ordering a decreasing chain of freely unequal words $W_1 > W_2 > W_3 > \cdots$ cannot be infinite. For a given group G we wish to extinguish all words $\varphi(S)$ to which another word $\psi(S)$ can be found which is equal to $\varphi(S)$ in G but precedes it in the sense of the ordering. These words $\varphi(S)$ form the domain of extinction of G. A set of reduced words which are all equal in G therefore contains a least element, the canonical representative of the group element. In particular, for Dyck words this is the 1. If R is a reduced word the set of elements of the form $x.R.v$ is called a ray with kernel R. The dot denotes that the factors on either side of it are not in mesh; otherwise they are arbitrary. If the kernel belongs to the domain of extinction then the whole ray does so as well. Let now R be an arbitrary Dyck word, split it in any way into two segments and reduce the segments to P and Q, respectively. If then $P > Q^{-1}$ the whole ray $x.P.v$ is to be extinguished and is called a left ray of extinction for the Dyck word R. The set of all left rays of extinction of R is called its left cluster of extinction. And if a cyclic Dyck word R is split into linear words in all possible manners then the set-theoretical sum of all its clusters is called the star of extinction of R. The corresponding right concepts are defined by reference to R^{-1}. The sum of left and right clusters (or stars) is the cluster (or star) $Z(R)$. In all these definitions R need not be a reduced Dyck word. It is shown, however, that if a reduction process is applied to R internally or externally, the rays of extinction can only increase and the clusters and stars actually remain the same. From this follows the theorem: the total domain of extinction of a group G is the sum of the stars of extinction of all its internally and externally reduced Dyck words. But in order to master the domain of extinction by actual construction of clusters (and stars) it is not necessary to know all rays of extinction. If R is split into the segments R' and R'' and $R' > R''^{-1}$ and there is no shorter segment of R with the same property, then the ray $x.R'.v$ is called a left fundamental ray of R. The knowledge of these rays is sufficient. Now it is easy, although it involves a great deal of combinatorial detail, to describe explicitly the kernels of all fundamental rays of a given internally and externally reduced word $R \approx a_1^{\alpha_1} \cdot \cdots \cdot a_p^{\alpha_p}$, where the a's are generators. Roughly speaking one must go "just beyond the middle" of R, but the details depend on whether p is odd or even, and on whether $a_1^{\alpha_1} > a_p^{-\alpha_p}$ or not. It must also be investigated how a possible correction of the order relations effects the domain of extinction. By this is meant the following process. Let the equality $S_i^n = 1$ be a consequence of the relations in G, but not of the formal order relations alone. If the generator S_i had formal order ∞, the additional relation $S_i^{|n|} = 1$ is introduced; if it had formal order n_i, then $S_i^{n_i} = 1$ is replaced by $S_i^{n_i'} = 1$, where $n_i' = (n_i, \mu)$. The effect of such a correction is fully elucidated.

(5) Another complication is introduced by the fact that if the link within which the word R is split belongs to a generator with infinite formal order, there is an infinity of distinct left fundamental rays. The author overcomes it by developing a theory of words in which the exponents of some of the generators (viz., those with infinite formal orders) are linear functions of a number of integer-valued arguments with integer coefficients. These are called z-words and the ordinary word arising from a given choice of the arguments a value of the z-word. A z-word is pseudo-reduced if adjacent powers belong to different generators and the exponents of the generators with finite formal orders lie in the appropriate $V'(n)$ and of the others are not identically zero. A value of a pseudo-reduced word may be reducible, but the author shows that to every pseudo-reduced z-word $W(z)$ a finite number (depending on the length of $W(z)$ only) of pseudo-reduced words $W^{(\lambda)}(z)$ can be found such that the irreducible values of these words are equal to all the values of $W(z)$. This remains true if the exponents in the z-word are subjected to a number of linear Diophantine inequalities provided that the values of the $W^{(\lambda)}(z)$ are also restricted by suitably related inequalities. The star of extinction of a z-word is the set of all stars of extinction of the values of the z-word. For a given pseudo-reduced z-word the star can again be constructed explicitly, naturally in this case with even more combinatorial detail which, however, does not involve more than the solution of linear equations and congruences.

(6) The decisive step towards the solution of the word problem is the transition from ordinary Dyck words to simple Dyck words. A Dyck word may still be of a very high degree of complexity. It may, e.g., possess a null-arc (i.e., an arc freely equal to 1), bounded by corresponding points of components of a freely inverse pair of factors. Any composition of factors which does not possess such a null-arc is called a simple composition. Now it is one of the general results on the process of composition: that if in a cyclic composition P two components W_1' and W_2' of the factors W_1 and W_2 are separated by a null-arc, then from the same factors as in P a new composition P' can be constructed which is freely equal to P, in which the components W_1' and W_2' are adjacent, and in which in the process of successive monotonic composition W_2 follows immediately upon W_1 or vice versa. A reduced Dyck word is called simple if it is freely equal to a simple composition of left hand sides of the relations or their inverses. The fundamental result is then: that the domain of extinction of a group G is already exhausted by the sum of the stars of extinction of its simple Dyck words, in other words that every star of extinction of a nonsimple Dyck word lies in the sum of the stars of extinction of simple compositions of factors of the form $f_i^{\alpha}(S)$. The proof of this main theorem is rather intricate and requires much space.

(7) The common feature of the cases I and II above is the following: every simple reduced Dyck word of a given length not exceeding Q can be obtained by a composition of the left hand sides of the defining relations and their formal inverses with a total reduced length of all factors L which is uniformly bounded by Q, viz., $L \leqq TQ$, where the constant T does not depend on Q, but only on the way in which the group G is given by generators and relations. In case I, $T=(1-8\delta)^{-1}\lambda_n/\lambda_1$; in case II, $T=9\lambda_n$. This boundedness makes it possible to tabulate, as it were, all simple reduced Dyck words by their length and leads to a solution of the word problem in the following manner. We wish to decide whether two words P and Q are equal in G, or whether the reduced linear word $R(S) \equiv P(S) \cdot Q(S)^{-1}$ is equal to 1 in G. If this is so, $R(S)$ has to lie in the domain of extinction of G. Let the length $l(R(S))=h$; then $R(S)$ must lie in the star of extinction of some simple cyclic Dyck word of a length not exceeding $2h+1$. All such words are values of a finite number of restricted cyclic s-words. Each of these can be brought to pseudo-reduced form. The required word $R(S)$ must then lie in one of the stars of extinction of a finite number of restricted s-words. Whether $R(S)$ belongs to these stars or not can be ascertained by reference to the explicit formulae. If not, then certainly $R \neq 1$. But if R falls into a ray of extinction of the star of one of these s-words, then from this s-word and the ray a reduced linear word $R_1(S)$ can be constructed with $R(S) > R_1(S)$ and $R_1 = R$ in G. The process is now repeated with R_1. But the decreasing chain $R(S) > R_1(S) > R_2(S) > \cdots$ cannot be infinite and must end with the canonical representative $R_k(S)$ of $R(S)$ in G. If $R_k(S) \equiv 1$, then P and Q are equal in G; otherwise they are not. It is clear, moreover, from the proof that in the case of equality a finite algorithm can be indicated (however difficult it may be to carry out in practice) of transforming P into Q by means of the defining relations of G.

K. A. Hirsch (Newcastle-upon-Tyne).

Wielandt, Helmut. Zur Theorie der einfach transitiven Permutationsgruppen. II. Math. Z. **52**, 384–393 (1949).

[For part I cf. Schur, S.-B. Preuss. Akad. Wiss. Phys. Math. Kl. **1933**, 598–623.] Given an abstract group H of order n it is asked under what conditions a primitive permutation group G of degree n can contain a regular subgroup isomorphic to H. I. Schur [loc. cit.] has treated the case when H is commutative. Treatment of the noncommutative case is begun in the present paper. The main result is that if H is a dihedral group of order greater than four then G is at least doubly transitive. *R. M. Thrall.*

Čunihin, S. A. On Π-properties of finite groups. Mat. Sbornik N.S. **25(67)**, 321–346 (1949). (Russian)

The paper gives a complete account, with proofs, of the author's investigations of Π-properties of finite groups. [For preliminary announcements and partial results see Rec. Math. [Mat. Sbornik] N.S. **4(46)**, 521–530 (1938); **15(57)**, 325–338 (1944); C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 43–45 (1943); **55**, 477–480 (1947); Doklady Akad. Nauk SSSR (N.S.) **58**, 1295–1296 (1947); **59**, 443–445 (1948); these Rev. **6**, 201; **5**, 143; **9**, 6, 492.] The basic aim is to establish, as far as possible, theorems in the abstract theory of finite groups which refer to a given set Π of prime numbers and are generalisations of the theorems which arise when the set Π is empty, consists of a single prime number, or contains all prime numbers. A typical example is Sylow's theorem which by a classical result of P. Hall [J. London Math. Soc. **3**, 98–105 (1928)] can be extended in the case of soluble groups to any division of the order into co-prime factors. Since this paper was written, further results in this field have been obtained by the author [Doklady Akad. Nauk SSSR (N.S.) **60**, 773–774 (1948); **66**, 165–168 (1949); these Rev. **10**, 589, 678] and by Gol'berg [ibid. **64**, 615–618 (1949); these Rev. **10**, 505]. *K. A. Hirsch.*

Hasse, Helmut. Verallgemeinerung des Dualitätssatzes für die Charaktere endlicher abelscher Gruppen auf beliebige endliche Gruppen. Math. Nachr. **3**, 1–3 (1949).

The author starts with the observation that the duality theorem for finite Abelian groups $G=\{S, \cdots\}$ implies that each nonzero representation $\chi \to a_\chi$ of the character group X of G by complex numbers is realized by a character, i.e., $a_\chi = a_\chi(S)$ for some $S \in G$. If G is an arbitrary finite group of g elements this remark leads to the following generalization which follows by direct verification from the orthogonality relations. Let $\Gamma_\chi = \{C_\chi(S)\}$ be a complete system of absolutely irreducible classes of representations of G of degrees f_χ belonging to the simple characters χ with the well-known formulas for the Kronecker products $\Gamma_\chi \times \Gamma^\psi = P_{\chi\psi}^{-1} \Gamma_{\chi\psi} P_{\chi,\psi}$, with representatives Γ_χ, where $\Gamma_{\chi\psi} = \operatorname{diag}\{I^{\xi}{}_{\chi\psi} \times \Gamma_\xi\}$, $\deg I^{\xi}{}_{\chi\psi} = (1/g)\sum s\chi(S)\psi(S)\xi(S^{-1})$ and a suitable matrix $P_{\chi,\psi}$. Then a nonzero system of matrices A_χ of degrees f_χ which is associated to the simple characters χ of G with a multiplication rule as the one given above (the Γ's replaced by the A's) is necessarily a set of representations, i.e., $A_\chi = C_\chi(S)$ for some $S \in G$. This is a special case of Tannaka's theorem [Tôhoku Math. J. (1) **45**, 1–12 (1938)]. *O. F. G. Schilling* (Chicago, Ill.).

Hasse, Helmut. Zur Frage der Zerfällungskörper des Gruppenrings einer endlichen Gruppe. Math. Nachr. **3**, 4–6 (1949).

The author proves the following consequence of the law of decomposition in cyclotomic fields $P_g^{(h)}$ of the g^{h+1}th roots of unity. If $\mathfrak{p}$ is a prime ideal of $P_g^{(h)}$ over $P_g^{(0)}$ then the local degree of $P_g^{(h)}$ at $\mathfrak{p}$ equals g^h/l_g where l_g does not depend on h, provided h is sufficiently large. This fact is used to validate the proof of the older result that the irreducible representations of a group of g elements lie in a field $P_g^{(h)}$ for sufficiently large h. For the complete proof of

Schur's conjecture see the work of R. Brauer [Amer. J. Math. 67, 461–471 (1945); these Rev. 7, 238].

O. F. G. Schilling (Chicago, Ill.).

Černikov, S. N. On complete groups with an ascending central series. Doklady Akad. Nauk SSSR (N.S.) 70, 965–968 (1950). (Russian)

This article is a continuation of work by the author [Rec. Math. [Mat. Sbornik] N.S. 18(60), 397–422 (1946); Mat. Sbornik N.S. 22(64), 319–348, 455–456 (1948); these Rev. 8, 311; 9, 566; 10, 10] and takes advantage of work of Malcev [Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 201–212 (1949); these Rev. 10, 507]. A group is complete if, for arbitrary n, every element is a product of nth powers. Let G be complete and possess an ascending central series [defined in the cited reviews]. If N_i are normal divisors of G, and if G/N_i have finite radical series ($i=1, 2$), then $G/N_1 \cap N_2$ has finite radical series; if G/N_i have finite rank and if G is moreover torsion-free, then $G/N_1 \cap N_2$ has finite rank. If G is torsion-free, and if some maximal Abelian normal divisor of G has finite rank, then G has finite rank.

J. L. Brenner (Pullman, Wash.).

Szele, Tibor. Zur Theorie der Zeroringe. Math. Ann. 121, 242–246 (1949).

An Abelian group G is said to be a nilgroup if the only ring R having additive group isomorphic to G is a zero ring (i.e., all products in R are zero). A group is said to be algebraically closed if for every integer n and every $\gamma \in G$ there exists $\xi \in G$ such that $n\xi=\gamma$. An Abelian group is said to be a torsion group if all its elements have finite order; it is said to be torsion free if zero is its only element of finite order; otherwise it is said to be mixed. It is proved (1) that a torsion group is a nilgroup if and only if it is algebraically closed; (2) that no mixed group is a nilgroup; (3) that every algebraically closed group is a direct sum of groups $Z(p^\infty)$, where p is a prime and $Z(p^\infty)$ is isomorphic to the additive group of rationals of the form a/p^e, $e=0, 1, 2, 3, \cdots$, taken mod 1.

R. M. Thrall (Ann Arbor, Mich.).

Kaloujnine, Léo. Addendum: Sur les groupes abéliens primaires sans éléments de hauteur infinie. C. R. Acad. Sci. Paris 230, 1327 (1950).

The author acknowledges that results analogous to his [same C. R. 225, 713–715 (1947); these Rev. 9, 268] were given by L. Kulikoff [Rec. Math. [Mat. Sbornik] N.S. 16(58), 129–162 (1945); these Rev. 8, 252].

Baer, Reinhold. Die Schar der Gruppenerweiterungen. Math. Nachr. 2, 317–327 (1949).

The purpose of the paper is to give a more conceptual form to the theory of group extensions and in particular, to free the theory from operating with residue class systems and factor sets. If γ is a homomorphism of a group G on a group M with the kernel E, the pair (G, γ) is an extension of M by means of E. To each such extension, there corresponds a homomorphic mapping K of G into the factor group of the group of all automorphisms of M modulo the group of inner automorphisms. In a paper which is to appear elsewhere, the author shows that every set of extensions of M by E with a given K can be imbedded in an extension (H, η) of a suitable group N by E. Here, N can be chosen as direct sum $M \oplus F$ of M with a free Abelian group F. Accordingly, in the present paper, the extensions of M by E are studied which are contained in a given extension (H, η) of a group $N \supseteq M$ by E.

Let (H, η) be an extension of a group N by E, and let M be a normal subgroup of H with $M \subseteq N$. A subgroup G of H is called an $M-E$-subgroup of H, if $H=N+G$, $M=N \cap G$. Such subgroups G exist, if and only if $(H/M, \eta)$ splits as an extension of N/M by E. If we denote the restriction of η to G again by η, then (G, η) becomes an extension of M by E. It is shown that if N/M is Abelian (as in the case $N=M \oplus F$), then the $M-E$-subgroups G of H form a commutative "Schar" in the sense of Prüfer. Here, for three $M-E$-subgroups X, Y, Z, the symbol $X-Y+Z$ is to denote the set of all elements $x-y+z$ with $x \in X$, $y \in Y$, $z \in Z$ and $x\eta=y\eta=z\eta$.

By an $M-E$-isomorphism α of an $M-E$-subgroup X on an $M-E$-subgroup Y, an isomorphism of X on Y is meant such that $x\alpha \equiv x \pmod N$ for $x \in X$ and that $x\alpha=x$ for $x \in M$. Similarly, an $N-E$-automorphism β of H is an automorphism of H such that $x\beta \equiv x \pmod N$ for $x \in H$ and that $x\beta=x$ for $x \in N$. It is shown that the $M-E$-isomorphism α of X on Y is induced by an $N-E$-automorphism of H if and only if $x\alpha-x$ belongs to the center $Z(N)$ of N for every $x \in X$. If the centralizer of M in N is contained in $Z(N)$, then every $M-E$-isomorphism of $X \to Y$ is induced by an $M-E$-automorphism of H. In particular, this will be so if N has the form $N=M \oplus F$ as above. Further, an abstract construction of the Schar $\Theta(K)$ of all classes of $M-E$-isomorphic extensions of M by E belonging to a given K (as defined above) is obtained [cf. R. Baer, Math. Z. 38, 375–416 (1934)]. Finally, it is shown that if E has finite order n, or if $nZ(M)=0$ for the center $Z(M)$ of M, then $n(X-Y)+Z=Z$ for $x, y, z \in \Theta(K)$, where the expression $n(X-Y)+Z$ is defined inductively in a natural manner.

R. Brauer (Ann Arbor, Mich.).

Baer, Reinhold. Klassifikation der Gruppenerweiterungen. J. Reine Angew. Math. 187, 75–94 (1949).

The author studies groups E which contain a fixed group U as a subgroup. It is not assumed that U is normal in E. Two extensions E and F of U in this sense are said to be equivalent if there exists a U-isomorphism of E on F, that is, an isomorphism which leaves U element-wise fixed. Two extensions E and F are almost-equivalent if there exist direct summands E' of E and F' of F which are equivalent extensions of U. Two extensions E and F are said to be related if they are direct summands of equivalent extensions. By a normal homomorphism of a group E on a group F, a homomorphism is meant which maps normal subgroups on normal subgroups. If E and F are extensions of U, and if there exist normal U-homomorphisms of E into F and of F into U, then E and F are said to be normal-similar. Finally, the extensions E and F of U are similar, if there exist U-homomorphisms of E into F and of F into E. Each of these five properties: "equivalent," "almost-equivalent," "related," "normal-similar," and "similar" implies the following one, but is actually narrower than it. It is shown that two extensions E and F are related if and only if there exists a U-homomorphism σ of E into F and a U-homomorphism τ of F into U such that $e\sigma$ commutes with $f-f\tau\sigma$ and that $f\tau$ commutes with $e-e\sigma\tau$ for every $e \in E$ and every $f \in F$. This implies that, in the case of Abelian groups E, F, similar extensions are related. Conditions are given under which related extensions are almost-equivalent. If $\Phi(U)$ is the collection of all classes of similar extensions (with due respect to the set-theoretical difficulties), a partial order can be introduced in $\Phi(U)$ by calling the class of E less than or equal to the class of F, if there exists a U-homomorphism

of E into F. Then every set of elements of $\Phi(u)$ has a least upper bound and a greatest lower bound in $\Phi(u)$. We say that a subgroup R is a retract of a group G, if there exist R-homomorphisms of G on R. If the class of E is less than or equal to the class of F, then there exists an extension of E which has F as a retract, and there exists an extension of E which has E as a retract and which can be mapped U-homomorphically on F. The converse of each of these two statements is true too. Two extensions E and F of U are similar if they are retracts of a common extension of E and F. The case of "normal" extensions (when U is normal in E and F) can be studied further. Finally the case of Abelian groups is considered. In all these results, the groups may be operator groups with a fixed domain of operators, if of course the concepts of subgroup and of homomorphism are suitably restricted. *R. Brauer.*

Lorenzen, Paul. Über halbgeordnete Gruppen. Math. Z. **52**, 483–526 (1949).

This paper extends to noncommutative groups certain basic portions of the theory of multiplicative ideal systems developed by the author in a previous paper [same Z. **45**, 533–553 (1939); these Rev. **1**, 101]. Three outstanding results are the following. (1) A lattice ordered group can be embedded in a vector group (cardinal product of linearly ordered groups) if and only if $a \wedge xax^{-1}=1$ implies $a=1$. (2) An r-group can be embedded in an r-allowable vector group if and only if its integral part is r-closed (definitions below). (3) A given group can be (linearly) ordered if and only if the unit subsemigroup $\{1\}$ is s-closed, a condition which involves only the group operation. By a semi-ordered group the author means a partially ordered group without the condition that $a \leqq b$ and $b \leqq a$ imply $a=b$. If the latter is satisfied, the semi-ordering is called strong; this is assumed in (3) above. As in the earlier paper, an ideal system $\mathfrak{S}_r$ of a semi-ordered group G can be defined by means of a mapping $\mathfrak{a} \to \mathfrak{a}_r$ of the set of finite subsets $\mathfrak{a}$ of G into the set of all subsets of G such that: (1) $\mathfrak{a} \subseteq \mathfrak{a}_r$; (2) $\mathfrak{a} \subseteq \mathfrak{b}_r$ implies $\mathfrak{a}_r \subseteq \mathfrak{b}_r$; (3) $\{a\}_r$ consists of all $x \geqq a$ in G; (4) $x\mathfrak{a}_r y \subseteq (x\mathfrak{a}y)_r$ for all x, y in G. Then $\mathfrak{S}_r$ consists of the sets $\mathfrak{a}_r$, which are called r-ideals. The s- and v-systems are defined as follows for any G, and $\mathfrak{a}_s \subseteq \mathfrak{a}_r \subseteq \mathfrak{a}_v$ for every r-system: $b \in \mathfrak{a}_v$ if and only if $a \leqq b$ for some $a \in \mathfrak{a}$; $b \in \mathfrak{a}_s$ if and only if $s \leqq xay$ for all $a \in \mathfrak{a}$ implies $s \leqq xby$. By an r-group is meant a semi-ordered group G in which a definite ideal system $\mathfrak{S}_r$ has been singled out. By this device the author achieves great generality, e.g., including the case G a field ordered by division relative to an integral domain $\mathfrak{g}$ within G, by taking for $\mathfrak{S}_r$ the d-system of all Dedekind ideals. Let G be an r-group, and $\mathfrak{g}$ the invariant subsemigroup thereof consisting of all integral elements a of G (i.e., all $a \geqq 1$). An invariant subsemigroup $\mathfrak{h}$ of G containing $\mathfrak{g}$ is called r-allowable if $\mathfrak{a} \subseteq \mathfrak{h}$ (finite $\mathfrak{a}$) implies $\mathfrak{a}_r \subseteq \mathfrak{h}$. If $x_1, \cdots, x_n \in G$, the intersection of all r-allowable subsemigroups of G containing $\mathfrak{g}$ and every x_i is denoted by $\mathfrak{g}(x_1, \cdots, x_n)_r$. An element y of G is called r-dependent on an r-ideal $\mathfrak{a}_r$ if there exist elements $x_1, \cdots, x_n$ of G such that $y \in \mathfrak{a}_r \cdot \mathfrak{g}(x_1^{\pm 1}, \cdots, x_n^{\pm 1})_r$ for each of the 2^n possible combinations of exponents ± 1. Defining $\mathfrak{a}_{r_s}$ to mean the set of all elements of G which are r-dependent on $\mathfrak{a}_r$, the mapping $\mathfrak{a} \to \mathfrak{a}_{r_s}$ (finite $\mathfrak{a}$) will define an ideal system $\mathfrak{S}_{r_s}$ if and only if $(1)_{r_s}=\mathfrak{g}$, in which case $\mathfrak{g}$ is called r-closed. Theorem (2) above can be expressed: $\mathfrak{g}$ is the intersection of linear r-allowable subsemigroups of G if and only if it is r-closed. A lattice ordered group $V_a^{(r)}$ containing G can be constructed without use of transfinite induction by taking the (dual) v-system of $\mathfrak{S}_{r_s}$. *A. H. Clifford.*

Michiura, Tadashi. On a definition of lattice ordered groups. J. Osaka Inst. Sci. Tech. Part I. **1**, 27 (1949).

It is noted in effect that the condition $(a+x-a)^*=a+x^*-a$ should be added to those of theorem 9 of the reviewer's paper on lattice-ordered groups [Ann. of Math. (2) **43**, 298–331 (1942), p. 302; these Rev. **4**, 3]. *G. Birkhoff.*

Montgomery, Deane. Connected two dimensional groups. Ann. of Math. (2) **51**, 262–277 (1950).

Le but de cet article est de prouver que tout groupe localement compact connexe séparable de dimension deux, qui n'est pas compact, est soit un groupe de Lie, soit le produit direct de la droite par un solénoide (le mot "dimension" est pris au sens de la théorie des espaces métriques séparables). La démonstration procède en plusieurs étapes. On montre tout d'abord, à l'aide de propriétés homologiques des espaces homogènes et d'un théorème de point fixe, qu'un groupe localement compact de dimension deux contient un noyau de sous-groupe à un paramètre; puis qu'un tel groupe G contient un noyau de sous-groupe C à deux paramètres (ce résultat se déduisant essentiellement du précédent et du fait qu'un espace fibré dont la base est un segment de droite est trivial, ainsi que du fait que tout groupe localement euclidien de dimension deux est de Lie); le sous-groupe H engendré par C est partout dense, et c'est l'image par un homomorphisme continu d'un groupe de Lie H^*. Si G n'est pas abélien, H^* est nécessairement le groupe $ax+b$; au sous-groupe de H^* formé des transformations $x \to x+b$ correspond alors dans G un sous-groupe invariant H_1 isomorphe à la droite; G/H_1 est de dimension un, donc ou bien isomorphe à la droite ou bien un solénoide, et le théorème annoncé se déduit sans grandes difficultés de ce fait. *R. Godement* (Nancy).

Gotô, Morikuni. A theorem on compact semi-simple groups. J. Math. Soc. Japan **1**, 270–272 (1949).

Let G be a connected compact topological group. The author shows that if the commutator subgroup of G is dense in G then it is actually the whole of G. In a final remark he states a related theorem about complex semisimple Lie groups which are not necessarily compact. According to this theorem the set of noncommutators is included in a set of complex dimension one less than that of G. *G. W. Mackey* (Cambridge, Mass.).

Gleason, A. M. Spaces with a compact Lie group of transformations. Proc. Amer. Math. Soc. **1**, 35–43 (1950).

Soient E un espace topologique et G un groupe topologique d'homéomorphismes de E sur E, de telle sorte que l'application $(s, p) \to s(p)$ de $G \times E$ dans E soit continue; pour tout $p \in E$, soit $G(p)$ l'ensemble des $s(p)$ $(s \in G)$; l'auteur cherche dans quelles conditions il existe dans E un ensemble fermé S qui rencontre chaque $G(p)$ en exactement un point, au moins pour les p qui sont suffisamment voisins d'un point donné de E ("sections locales"). Le principal résultat est le suivant: supposons E complètement régulier, G compact de Lie, et soit p_0 un point tel que $s(p_0)=p_0$ implique $s=e$; alors il existe une section locale au voisinage de p_0. La démonstration s'effectue comme suit. Tout d'abord, soit $s \to H_s$ une représentation linéaire fidèle de G, de dimension n; les H_s forment un sous-groupe fermé du groupe de Lie $GL(n)$, donc défini par des équations différentielles bien connues; de là résulte l'existence dans $GL(n)$ d'un voisinage V de 1 et d'une section locale S_0 de V relativement aux classes à droite modulo le sous-groupe des H_s. Maintenant, par une extension facile du théorème d'Urysohn, on peut

construire sur E une fonction continue $f(p)$, à valeurs dans l'algèbre M_n des matrices d'ordre n, et vérifiant $f(s(p_0))=H_s$ pour tout $s\in G$ (ceci étant permis du fait que G agit simplement sur la trajectoire de p_0); en remplaçant au besoin f par la fonction $\int_G H_s^{-1}\cdot f(s(p))ds$ (ds mesure de Haar sur G), on peut supposer en outre réalisée la condition $f(s(p))=H_s\cdot f(p)$; soit alors S' l'ensemble des $p\in E$ tels que $f(p)\in S$; de la relation précédente résulte que la trajectoire de tout $p\in E$ assez voisin de p_0 rencontre S'; et rencontre S' en un seul point, car f étant continue et vérifiant $f(p_0)=1_n$ est inversible pour p assez voisin de p_0, de sorte que, p étant donné, la relation $H_s\cdot f(p)\in S$ détermine entièrement s: d'où le théorème.

L'auteur étend ensuite le théorème précédent au cas où la condition suivante est réalisée: G n'agit pas simplement sur l'orbite de p_0, mais, pour tout p assez voisin de p_0, le stabilisateur G_p de p dans G est conjugué de G_{p_0}; finalement, le théorème que voici est démontré: soient E un groupe topologique quelconque et G un sous-groupe fermé de G, qui est un revêtement d'un groupe de Lie compact; alors il existe au voisinage de l'unité une section relativement aux classes à droite modulo G. *R. Godement* (Nancy).

Gel'fand, I. M. The center of an infinitesimal group ring. Mat. Sbornik N.S. **26(68)**, 103–112 (1950). (Russian)

Le but de cet article est d'exposer quelques propriétés élémentaires des algèbres associatives engendrées par une algèbre de Lie. Soit G un groupe de Lie, R l'algèbre de Lie correspondante, K l'algèbre associative (de dimension infinie) engendrée par R de façon évidente; si (e_i) est une base de R, tout élément de K peut se mettre sous la forme $\alpha\cdot I+\alpha^i e_i+\alpha^{ij}e_ie_j+\cdots$ (on adopte la convention d'Einstein); si R' est l'espace vectoriel dual de R, on peut associer à un tel élément les formes multilinéaires $\alpha^i\xi_i$, $\alpha^{ij}\xi_i\eta_j$, $\cdots$ (ξ_i, η_j, $\cdots$ sont les coordonnées, relativement à la base duale de e_i, d'éléments variables de R'); l'auteur démontre tout d'abord que pour que l'élément donné de K soit dans le centre de K, il suffit que les formes multilinéaires correspondantes soient invariantes par le groupe adjoint de G; cette condition est de plus nécessaire si l'on suppose (ce qui est permis grâce aux relations de commutation des e_i) que les coefficients α^{ij}, $\cdots$, sont symétriques. Exemple: si G est le groupe $GL(n, R)$, et si e_i^j est la base bien connue de l'algèbre de Lie correspondante, le centre de K est isomorphe à l'algèbre des polynômes par rapport aux n éléments e_i^i, $e_i^je_j^i$, $\cdots$, $e_{i_1}^{i_1}e_{i_2}^{i_2}\cdots e_{i_1}^{i_n}$; l'auteur donne un résultat analogue pour le groupe orthogonal. Au lieu de considérer les formes multilinéaires sur R', invariantes par le groupe adjoint, on peut bien entendu considérer les polynômes symétriques formés avec les coordonnées d'un vecteur variable de R'.

L'utilité de considérer le centre de l'algèbre K provient du fait évident que, dans toute représentation irréductible de dimension finie de K, le centre de K est appliqué sur les scalaires; l'auteur illustre ce fait à l'aide des espaces de tenseurs dans le cas de $GL(n, R)$.

[Il y a lieu de confronter cet article avec un travail récent de Harish-Chandra sur un sujet analogue, Ann. of Math. (2) **50**, 900–915 (1949); ces Rev. **11**, 77.]

R. Godement (Nancy).

Gel'fand, I. M. Spherical functions in symmetric Riemann spaces. Doklady Akad. Nauk SSSR (N.S.) **70**, 5–8 (1950). (Russian)

Soient G un groupe de Lie, H un sous-groupe compact de G, X l'espace des classes à droite Hg modulo H. D'autre part, dans l'espace L^1 construit sur une mesure de Haar de G, considérons l'ensemble R des fonctions constantes sur les classes bilatères modulo H; c'est évidemment une sous-algèbre fermée de L^1 (pour le produit de composition), invariante par l'involution $f(g)\to\overline{f(g^{-1})}$; les formes linéaires positives sur R correspondent, de façon evidente, aux fonctions $\varphi(g)$, continues, de type positif, constantes sur les classes bilatères modulo H; cette correspondance transforme toute forme élémentaire (relativement à R) en une fonction élémentaire (relativement à G); de plus, les représentations unitaires de G définies par de telles $\varphi(g)$ sont caractérisées par le fait de contenir des vecteurs non nuls invariants par H; enfin, à une fonction $\varphi(g)$ du type précédent correspond sur l'espace X un "noyau" $K(x_1, x_2)=\varphi(g_2^{-1}g_1)$, qui est défini positif et invariant par les "déplacements" de X définis par les éléments de G; les noyaux K correspondants à des fonctions φ élémentaires sont appelés les fonctions sphériques zonales de X.

Supposons maintenant qu'il existe sur G un automorphisme involutif conservant les éléments de H; X est alors un espace de Riemann symétrique [en fait, il faut supposer que H est l'ensemble de tous les éléments invariants par cet automorphisme [cf. É. Cartan, La théorie des groupes finis et continus et l'analysis situs, Mémor. Sci. Math., no. 42, Gauthier-Villars, Paris, 1930]; cette condition est certainement nécessaire aussi pour assurer ce qui suit]; l'auteur affirme qu'alors l'anneau R est commutatif, de sorte qu'alors la structure des formes positives sur R est facile à élucider: tout noyau défini positif invariant par "déplacement" est une somme continue de fonctions sphériques zonales.

Soit S l'espace des classes bilatères modulo H; on peut identifier alors R à un anneau de fonctions définies sur S, le produit dans S étant défini par une formule du type $f_1*f_2(s)=\iint f_1(s_1)f_2(s)a(s_1, s_2, s)ds_1ds_2$ où la fonction a et la mesure ds sont bien déterminées sur S, et ont du reste une signification géométrique simple. Dans des hypothèses telles que R soit commutatif les fonctions sphériques zonales vérifient alors l'équation fonctionnelle

$$\int a(s_1, s_2, s)\varphi(s)ds=\varphi(s_1)\cdot\varphi(s_2).$$

[Note du rapporteur: tout ceci est en relation étroite avec les "systèmes de translations généralisées" de Levitan [cf. par exemple Uspehi Matem. Nauk (N.S.) **4**, no. 1(29), 3–112 (1949); ces Rev. **11**, 116] et n'est qu'un cas particulier de théorèmes valables dans tout système de translations généralisées commutatif; il est quelque peu surprenant que l'auteur ne fasse aucune allusion aux travaux de Levitan, lesquels s'appliquent, non seulement à la situation présente, mais aussi, comme il est facile de le voir, à l'étude des fonctions centrales sur les groupes de Lie compacts: ces fonctions forment en effet un système de translations généralisées si on les considère sur un sous-groupe abélien maximal.]

Soit G un groupe de Lie; on peut considérer son algèbre de Lie L, puis l'algèbre associative engendrée par L, soit A; les éléments de A définissent de façon évidente des opérateurs différentiels sur G, invariants par les translations à gauche; le centre de A conduit à des opérateurs invariants à gauche et à droite; si $G=Gl(n)$, ces derniers opérateurs contiennent n éléments algébriquement indépendants [cf. l'analyse précédente]; dans tous les cas, ces opérateurs définissent sur l'espace symétrique X des "opérateurs de Laplace" généralisés, dont les fonctions sphériques zonales sont des solutions propres: pour la simple raison que, dans toute représentation unitaire irréductible de G, les éléments

du centre de A sont (lemme de Schur) appliqués sur des scalaires; l'auteur annonce que, dans le cas des groupes semi-simples, il n'existe qu'un nombre fini d'opérateurs de Laplace indépendants, et que les fonctions sphériques zonales sont caractérisées par le fait d'être des solutions propres de ces opérateurs, et d'être invariantes par les déplacements de X.

Note du rapporteur: les résultats précédents permettent de décomposer les représentations unitaires de G pour lesquelles il existe "suffisamment" de vecteurs non nuls invariants par H; pour aborder le cas général, il faudrait faire une étude des fonctions $\varphi(g)$ telles que, pour un caractère donné $\chi(u)$ du sous-groupe H, on ait $\varphi(g)=\int_H \varphi(gu)\cdot\chi(u)du$ et $\varphi(ug)=\varphi(gu)$; il est probable que les sous-algèbres de L^1 ainsi obtenues, bien que non commutatives en général, sont d'une étude relativement aisée; de plus, on serait conduit ainsi à des fonctions "sphériques" d'ordre supérieur: c'est en tous cas ce qui se passe dans le groupe hyperbolique par exemple.

R. Godement (Nancy).

*Stiefel, E. **Sur les nombres de Betti des groupes de Lie clos.** Topologie algébrique, pp. 97–101. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12. Centre de la Recherche Scientifique, Paris, 1949. 600 francs.

The author indicates briefly the principle of a method which leads to the determination of the Betti numbers of the classical groups. The method is based on the study of the discrete group whose fundamental domain is the polyhedron of Cartan. Its applicability seems to depend on the existence in a simple group of rank l (compact, simply connected) of a subgroup of rank $l-1$ which is not homologous to 0; if this is the case, then the method is certainly not applicable to all exceptional groups, since it can be shown that every closed subgroup of rank 7 of E_8 is homologous to 0.

C. Chevalley (New York, N. Y.).

de Siebenthal, Jean. Sur certaines sous-groupes de rang un des groupes de Lie clos. C. R. Acad. Sci. Paris **230**, 910–912 (1950).

Let G be a compact semi-simple Lie group of rank l. Certain rays d lying in the fundamental polyhedral angle of G correspond to subgroups g of G of rank 1. The author determines those rays d such that the corresponding g's of rank one are not contained in proper subgroups of G of rank l. Among such g's there is one, called principal, which is unique up to conjugation in G. The principal g corresponds to a certain principal "diagonal" ray.

P. A. Smith (New York, N. Y.).

Ważewski, T. Exemples des groupes de transformations d'une droite en elle-même qui dépendent de quatre paramètres essentiels. Prace Mat.-Fiz. **47**, 105–116 (1949).

S. Lie has proved that a Lie group of analytic transformations of the real line into itself is of dimension not exceeding 3. The author shows by an example that this is not true any more for a Lie group of transformations of class C^∞. The idea of the construction is as follows. Given any $m>0$, we can find m open sets Ω_i $(1\leqq i\leqq m)$ which are mutually disjoint and whose adherences all contain some point ξ^*. For each Ω_i we can find a one-parameter group G_i of transformations of class C^∞ whose operations (other than the identity) leave fixed the points of the complement of Ω_i, and only these. The group G generated by $G_1, \cdots, G_m$ is then an m-dimensional group of transformations of class C^∞ of the straight line into itself. The sets Ω_i may be selected so as to all have a given totally discontinuous closed set F in their adherence; the group G will then be "locally m-dimensional" (in an easily defined sense) at each point of F. The author raises the question as to whether the set of points at which a Lie group G of transformations of class C^∞ of the real line is locally of dimension at most 3 is dense; the answer to this question is affirmative. *C. Chevalley.*

Clifford, A. H. Extensions of semigroups. Trans. Amer. Math. Soc. **68**, 165–173 (1950).

This paper is concerned with the extension problem for semigroups, i.e., the determination of all semigroups Σ with a given semigroup S as an ideal and such that $\Sigma-S$ is isomorphic to another given semigroup T, T necessarily having a zero. If S has an identity element then all extensions of S are determined by ramified homomorphisms of T into S, a ramified homomorphism of T into S being a mapping $A\to\bar{A}$ of the set T of nonzero elements of T into S with the property that, if $AB\neq 0$, then $\overline{AB}=\bar{A}\bar{B}$. The author then considers the case where S is restricted to satisfy the following condition, weaker than the existence of an identity; if a, b are two distinct elements of S, then there exists an element s of S for which either $as\neq bs$ or $sa\neq sb$. In this case, every extension is determined by a pair of linked ramified homomorphisms of T into the semigroups of left and right translations of S. (A left translation λ of S is a mapping $x\to\lambda x$ of S into itself satisfying the condition $\lambda(xy)=(\lambda x)y$, while a right translation ρ is a mapping $x\to x\rho$ of S into S satisfying $(xy)\rho=x(y\rho)$; λ and ρ are linked if $(x\rho)y=x(\lambda y)$ identically in x, y.) Finally the author applies this theory to determine the extensions of completely simple semigroups [Rees, Proc. Cambridge Philos. Soc. **36**, 387–400 (1940); these Rev. **2**, 127].

D. Rees (Cambridge, England).

NUMBER THEORY

Srinivasan, M. S. Theory of pharinxes. J. Univ. Bombay (N.S.) **18**, part 3, sect. A, 4–22 (1949).

The author states a number of known results on the binomial congruence. These are couched in his own terminology and notation and are difficult to decipher.

D. H. Lehmer (Berkeley, Calif.).

Palamà, Giuseppe. Tabella delle posizioni iniziali relative al "Neocribrum" di L. Poletti. Rivista Mat. Univ. Parma **1**, 85–98 (1950).

The table gives for each prime p $(17\leqq p\leqq 3547)$ the solutions x of the congruences $30(x_i-1)+m_i\equiv 0 \pmod p$ where m_i are the 6 numbers less than and prime to 30. The least positive residue of -1001 modulo p is also given. The table is intended to assist any one wishing to construct a factor table based on the modulus 30030. *D. H. Lehmer.*

Palamà, Giuseppe. Somma termine a termine e sequenze di multigrade. Partizione dei numeri. Multigrade a catena. Applicazioni. Atti Sem. Mat. Fis. Univ. Modena **3**, 162–190 (1949).

In this paper, which is divided into five parts, methods of combining solutions of multigrade equations are considered. Thus in part I it is shown under what conditions,

when $n=2$, 3 or 5, two solutions

$$a_1, \cdots, a_p \stackrel{n}{=} b_1, \cdots, b_p,$$
$$a_1', \cdots, a_p' \stackrel{n}{=} b_1', \cdots, b_p'$$

may be combined to form a solution

$$ma_1+m'a_1', \cdots, ma_p+m'a_p' \stackrel{n}{=} mb_1+m'b_1', \cdots, mb_p+m'b_p',$$

m, m' being arbitrary integers. In part II a generalized Tarry's theorem [see the author, Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 6, 95–120 (1947); these Rev. 9, 9] is used to partition the first p integers into multigrade equation chains. Other methods of obtaining chains are discussed in part III. The methods of part III are used in part IV to construct an 8×8 magic square and a 4×8 magic rectangle using the integers $1, 2, \cdots, 64$ and $1, 2, \cdots, 32$ respectively. Finally, in part V, sequences of solutions of multigrade equations are discussed. The paper contains many numerical examples to illustrate the theory.

W. H. Simons (Kingston, Ont.).

Moessner, Alfred. On equal sums of powers. Math. Student 15 (1947), 83–88 (1949).

This paper gives a parametric solution, with many examples, of the Tarry-Escott problem including two sets of 20 linear functions in 6 variables having the same sum of nth powers for $n=0, 1, \cdots, 10$. *D. H. Lehmer.*

Gloden, A. Impossibilités diophantiennes. Euclides, Madrid 9, 476 (1949).

Si l'on a $2x^2+1=y^2$, pour cette même valeur de x l'équation $3x^2+1=z^2$ est impossible, sauf au cas trivial où $x=0$; alors $y=z=\pm1$. *Extract from the paper.*

Vivanti, Giulio. Un teorema di aritmetica e la sua relazione colla ipotesi di Fermat. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 11(80) (1947), 239–246 (1949).

The author proves the known facts that if $(x, y)=1$, $x\pm y$ and $(x^p\pm y^p)/(x\pm y)$ are relatively prime if $p\mid(x\pm y)$ and that if $p\mid(x\pm y)$ then p, but not p^2, divides $(x^p\pm y^p)/(x\pm y)$. From this theorem he derives several observations concerning Fermat's last theorem, e.g., if $(x, y)=1$ and $x\pm y$ is a product of different odd primes, then there is no odd m for which $x^m\pm y^m$ is equal to an mth power. He concludes that the probability of the failure of Fermat's last theorem is very small. *N. G. W. H. Beeger* (Amsterdam).

Inkeri, K. On the second case of Fermat's last theorem. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 60, 32 pp. (1949).

The author extends results of Vandiver on the equation (1) $\alpha^l+\beta^l+\gamma^l=0$, where l is an odd prime. Many of the theorems contain complicated assumptions about the nature of primes p for which the congruence $x^l+y^l+z^l\equiv0 \pmod p$ has no solution in integers x, y, z, prime to p. These are employed to prove that if $l\leqq617$ then equation (1) is impossible if α, β, γ are integers, none zero, in the field $K(\zeta+1/\zeta)$, where $\zeta=\exp(2\pi i/l)$. *D. H. Lehmer.*

Salié, Hans. Über den kleinsten positiven quadratischen Nichtrest nach einer Primzahl. Math. Nachr. 3, 7–8 (1949).

Linnik has proved that if l and k are coprime positive integers, then the smallest prime number congruent to l modulo k does not exceed k^c, where c is a positive absolute constant [Rec. Math. [Mat. Sbornik] N.S. 15(57), 139–178, 347–368 (1944); these Rev. 6, 260]. From this result the author deduces that the smallest quadratic nonresidue modulo an odd prime p is not $o(\log p)$ as p tends to infinity. Independently Fridlender [Doklady Akad. Nauk SSSR (N.S.) 66, 351–352 (1949); these Rev. 10, 684] has already remarked that Linnik's theorem has this corollary.

P. T. Bateman (Urbana, Ill.).

Specht, Wilhelm. Zahlenfolgen mit endlich vielen Primteilern. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1948, 149–169 (1949).

Let $\mathfrak{B}$ denote the sequence $1=b_1<b_2<\cdots$ of integers which contain no prime factors other than the m given primes $p_1, \cdots, p_m$ $(m\geqq2)$, and let $B(x)$ be the number of members of $\mathfrak{B}$ which do not exceed x. Put $P=(p_1p_2\cdots p_m)^{\frac{1}{2}}$, $Q=(m!\log p_1\log p_2\cdots\log p_m)^{1/m}$. The author improves on certain results given by G. Pólya and himself concerning the sequence $\mathfrak{B}$. Thus he proves that, for large n and x,

$$\log b_n = Qn^{1/m}-\log P+o(1), \tag{1}$$

$$B(x) = \{(\log x)/Q\}^m+o\{(\log x)^{m-1}\}, \tag{2}$$

$$b_{n+1}\sim\frac{1}{P}\exp(Qn^{1/m}), \tag{3}$$

$$\log(b_{n+1}-b_n)\sim\log b_n\sim Qn^{1/m}. \tag{4}$$

In order to prove (1), (2) and (3) he considers the set $\Delta_m(\xi, \alpha)$ of points $(x_1, x_2, \cdots, x_m)$ with nonnegative integral coordinates which satisfy the inequality $\sum_{\nu=1}^m\alpha_\nu x_\nu\leqq\xi$, where $\xi>0$ and $\alpha_\nu>0$ $(\nu=1, \cdots, m)$. The number of points in $\Delta_m(\xi, \alpha)$ he calls $A_m(\xi, \alpha)$ and proves that, if the ratios α_ν/α_1 are not all rational then

$$A_m(\xi, \alpha) = \frac{\{\xi+\frac{1}{2}(\alpha_1+\alpha_2+\cdots+\alpha_m)\}^m}{m!\alpha_1\alpha_2\cdots\alpha_m}+o(\xi^{m-1}) \tag{5}$$

as $\xi\to\infty$. Relation (5) is deduced from some general results on uniform distribution of functions $f(x, \alpha)$ with respect to $\Delta_m(\xi, \alpha)$ and is applied to evaluate $B(x)$ and b_n by taking $\xi=\log x$, $\alpha_i=\log p_i$ $(1\leqq i\leqq m)$. The proof of (4) requires the Thue-Siegel theorem which is used to show that the Diophantine inequalities $0<Ax^p-By^p\leqq Cy^{p-2\sqrt{p}}$ (A, B positive integers, $p\geqq5$ a prime and $C>0$ an arbitrary constant) has only a finite number of solutions. By putting $b_{n+1}=Ax^p$, $b_n=By^p$ where A and B are divisible by no pth powers, it is deduced that the inequality $b_{n+1}-b_n\leqq b_n^{1-\epsilon}$ $(\epsilon>0)$ can hold for a finite number of values of n only and (4) follows.

R. A. Rankin (Cambridge, England).

Yakovkin, M. V. Polynomials taking values with a bounded number of prime divisors. Doklady Akad. Nauk SSSR (N.S.) 69, 739–742 (1949). (Russian)

The author gives a number of criteria for the irreducibility of polynomials with integral coefficients, of which the following theorem is one. Let $f(x)$ be a polynomial of degree n the highest common factor of whose coefficients is unity and suppose that $x_1, x_2, \cdots, x_r$ are integers incongruent modulo p (a prime) such that, if α is the real part of any root of $f(x)=0$, then $|\alpha-x_i|>q$ for $i=1, 2, \cdots, r$ and for each α. Also suppose that, for each i, $f(x_i)$ is of the form p^kt where $t\leqq q^s$. Then $f(x)$ cannot possess a factor with coefficients in the field of rational numbers and of degree m where $s\leqq m<r$. From this theorem, and from other similar results, certain criteria similar to those given by A. Brauer and R. Brauer [Math. Z. 40, 242–265 (1935)] and J. O.

Reynolds [J. Elisha Mitchell Sci. Soc. 63, 120–132 (1947); these Rev. 9, 331] can be deduced as special cases.

R. A. Rankin (Cambridge, England).

Duarte, F. J. Monograph on the numbers π and e. Historical and bibliographical notes. Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. 11 (1948), no. 34–35, 1–252 (1949). (Spanish)

This monograph probably contains more information on π and e than has ever before been collected in one place. There are historical details; a large collection of numerical and geometrical approximations, formulas involving π and e, and decimal values of functions of these numbers; no less than five (classical) proofs of the transcendence of e and π; and even mnemonics, paradoxes and jokes. Some of the formulas and numerical values seem to be new.

R. P. Boas, Jr. (Providence, R. I.).

Brauer, Alfred, and Macon, Nathaniel. On the approximation of irrational numbers by the convergents of their continued fractions. II. Amer. J. Math. 72, 419–424 (1950).

Let ξ be a positive irrational number with the regular continued fraction expansion $\xi=[q_0, q_1, \cdots]$ and convergents A_n/B_n. Define λ_n by the equation $|\xi-A_n/B_n|=1/(\lambda_n B_n^2)$. A portion of part I of this paper [same J. 71, 349–361 (1949); these Rev. 10, 513] was concerned with lower bounds for sums of two, three, four and eight consecutive λ's, and these results were used to obtain a lower bound for (*) $\liminf (\sum_0^{m-1}\lambda_n)/m$. In the present paper best possible lower bounds are obtained for the sum of k consecutive λ's, and from this the best possible lower bound for (*) is obtained. Let $f_{-1}, f_0, f_1, \cdots$ be the Fibonacci sequence in which $f_{-1}=0$, $f_0=1$, $f_{i+1}=f_i+f_{i-1}$ for $i\geqq 0$, and let $F_k=\sum_{i=0}^{k-1} f_{i-1}/f_i$. Then it is shown that, for every ξ, $\sum_{i=0}^{k-1}\lambda_{n+i}>k+2F_k$ for every $k>0$ and for every n, and it follows from this that (*) is not smaller than $\sqrt{5}$. These are best possible bounds in the following sense: for every $k>0$ and every $\epsilon>0$ there is a ξ such that, for infinitely many values of n, the last written sum is smaller than $k+2F_k+\epsilon$.

W. J. LeVeque (Ann Arbor, Mich.).

Steuerwald, Rudolf. Bemerkungen zu einer Arbeit von Herrn J. Heinhold. Math. Z. 52, 394–400 (1949).

The author settles some of the undecided cases in Heinhold's tables [Math. Z. 44, 659–688 (1939)], relating to the minimum of the quadratic form x^2-Dy^2 or $x^2+xy-\frac{1}{4}(D-1)y^2$ for $x\equiv x_0 \pmod 1$, $y\equiv y_0 \pmod 1$.

H. Davenport.

Prachar, K. Über höhere Minima quadratischer Formen. Monatsh. Math. 53, 268–277 (1949).

The author defines M_k for $k=1, 2, \cdots$ to be the least number such that the binary quadratic form $ax^2+2bxy+cy^2$, with real coefficients satisfying $ac-b^2=1$, assumes values not exceeding M_k for at least k pairs of integers (x, y) and $(-x, -y)$, excluding $(0, 0)$. He proves that if k is greater than a certain absolute constant, then $M_k=k^2(k^2-\frac{1}{4})^{-1}$. He also evaluates $M_2\ (=8/\sqrt{15})$, and the constant analogous to M_2 for binary Hermitian forms with variables in certain particular quadratic fields.

H. Davenport (London).

Rogers, C. A. The product of n real homogeneous linear forms. Acta Math. 82, 185–208 (1950).

Let $x_1, \cdots, x_n$ be n real homogeneous linear forms in n integral variables $u_1, \cdots, u_n$, with determinant unity. The lower bound of $|x_1x_2\cdots x_n|$ for all $(u_1, \cdots, u_n)$ other than $(0, 0, \cdots, 0)$ depends upon the coefficients of the forms and has an upper bound M_n. Let $M=\limsup_{n\to\infty}(M_n)^{1/n}$. It is shown that $M\leqq\frac{1}{4}\pi e^{-1}=1/5.70626\cdots$. This estimate is an improvement on an earlier result of the author [J. London Math. Soc. 24, 31–39 (1949); these Rev. 10, 512] and also on a result $M\leqq(2\pi)^{-1}e^{-1}=1/5.30653\cdots$ which he shows to be implicit in a paper of Blichfeldt. The proof is based upon a lemma which gives a sharper estimate for the upper bound of $K_m=\prod|z_r-z_s|^{2/(m(m-1))}/\{\sum|z_t|/m\}$ $(1\leqq r<s\leqq m, 1\leqq t\leqq m, z_t$ real) than that given in the earlier paper. It is for large m that this estimate is required, and what the author does is, effect, to approximate to K_m by the expression

$$K=\exp\left\{\int_0^1 dx\int_0^1 \log|\alpha(x)-\alpha(\xi)|\,d\xi\right\}\Big/\int_0^1|\alpha(x)|\,dx,$$

where $\alpha(x)$ is an increasing function satisfying certain conditions. The determination of the function $\alpha(x)$ which makes K a maximum is a problem in the calculus of variations which the author (who makes acknowledgment to L. A. Wigglesworth for help) solves; in fact, $\max K=\frac{1}{4}\pi e^{-1}$.

R. A. Rankin (Cambridge, England).

Chalk, J. H. H. On the product of non-homogeneous linear forms. J. London Math. Soc. 25, 46–51 (1950).

Let $L_1, \cdots, L_n$ be arbitrary real linear forms in integral variables $u_1, \cdots, u_n$ with determinant 1, and let $c_1, \cdots, c_n$ be arbitrary real numbers. Let γ_n be the lower bound of the numbers γ such that there always exist integers $u_1, \cdots, u_n$ satisfying (*) $|(L_1+c_1)\cdots(L_n+c_n)|\leqq\gamma$. By a result of Davenport, which is a refinement of an earlier result of Tschebotareff, we know that $\gamma_n\leqq\gamma_n'$ where $\gamma_n'<(\sqrt{2})^{-n}$ and $\gamma_n'\sim(2e-1)^{-1}(\sqrt{2})^{-n}$ as $n\to\infty$ [H. Davenport, same J. 21, 28–34 (1946); 24, 316 (1949); these Rev. 8, 443; 11, 233, and references given there]. The author supposes that $\gamma>\gamma_n'$ and that none of the forms $L_1+c_1, \cdots, L_n+c_n$ assumes the value zero, and he proves that the inequality (*) has an infinity of sets of integral solutions if and only if the forms $L_1, \cdots, L_n$ are inexpressible in the form

$$L_r=\lambda_r(a_{r1}u_1+\cdots+a_{rn}u_n), \qquad r=1, \cdots, n,$$

where the a's are integers and the λ's are real.

C. A. Rogers (Princeton, N. J.).

Ricci, Giovanni. Figure, reticoli e computo di nodi. Rend. Sem. Mat. Fis. Milano 19 (1948), 165–205 (1949).

This is an expository and historical account of the classical lattice-point problems, including the recent work of D. G. Kendall [Quart. J. Math., Oxford Ser. 19, 1–26 (1948); these Rev. 9, 570]. The regions considered include the square, rhombus, circle, hyperbola (divisor problem) and multidimensional ellipsoid.

R. A. Rankin.

Shapiro, Harold N. An elementary proof of the prime ideal theorem. Comm. Pure Appl. Math. 2, 309–323 (1949).

In this paper the "elementary" methods used by Selberg to prove the prime number theorem [Ann. of Math. (2) 50, 305–313 (1949); these Rev. 10, 595] are extended to yield a proof of the prime ideal theorem. This theorem asserts that $\theta_K(x)\sim x$, for large x, where K is a fixed algebraic number field,

$$\theta_K(x)=\sum_{N\mathfrak{p}}\log N\mathfrak{p},$$

and $\mathfrak{p}$ denotes a prime ideal. The proof is based upon the

following analogue of Selberg's identity, namely

$$\sum_{N_1 N_2 \leq x} \log^2 N_1 + \sum_{N_1 N_2 \leq x} \log N_1 \log N_2 = 2x \log x + O(x).$$

Just as in Selberg's paper an iterative process is set up to estimate the error $R_K(x) = \theta_K(x) - x$ and it is deduced that $R_K(x) = o(x)$. The author states that his theorem implies the prime number theorem for arithmetic progressions and concludes with a generalization to free Abelian groups.

There are places in the text where the argument is not clear. Thus, on page 317 (case (b)), it is not explained why the inequality $|R_K(y)| \leqq |R_K(y) - R_K(y-1)| - 1$ holds; it would appear to be false if, for example,

$$-\tfrac{1}{2} < R_K(y) < 0 < R_K(y-1) < \tfrac{1}{2}.$$

Further, it seems to the reviewer that, if lemma 3.3 is applied in the way indicated at the crux of the proof on page 321, the inequalities $\rho^i \leqq y_i \leqq \rho^{i+1} e^{\frac{1}{2}}$ should be replaced by $x\rho^{-i-1} \leqq y_i \leqq x\rho^{-i} e^{\frac{1}{2}}$; from this the required inequality for $|R_K(x)| \log^2 x$ can be deduced by a route different from that given. *R. A. Rankin* (Cambridge, England).

Mirsky, L. A theorem on sets of coprime integers. Amer. Math. Monthly **57**, 8–14 (1950).

Let n, s, r be integers such that $n \geqq 1$, $2 \leqq r \leqq s$, and let $N_{s,r}(n)$ denote the number of representations (order being relevant) of n as the sum of s positive integers such that the highest common factor of any r of them is equal to 1. A complicated asymptotic formula, as $n \to \infty$, is given for $N_{s,r}(n)$. In the case $r = s$, the result is

$$N_{s,s}(n) = \frac{n^{s-1}}{(s-1)!} \prod_{p \mid n} (1 - p^{-s+1}) + O(n^{s-2}).$$

The true order of magnitude of $N_{s,r}(n)$ is shown to be n^{s-1}. *W. H. Simons* (Kingston, Ont.).

Richert, Hans-Egon. Über Zerfällungen in ungleiche Primzahlen. Math. Z. **52**, 342–343 (1949).

The author shows that every positive integer greater than 6 is the sum of distinct primes. In fact, the 13 numbers from 7 to 19 inclusive are sums of distinct primes less than 13. If we add 13 to these numbers and 17 to the resulting sums, etc., we obtain the theorem by a process which cannot fail, since there is always a prime between x and $2x$ for $x \geqq 2$. [Compare R. Sprague, same Z. **51**, 289–290, 466–468 (1948); these Rev. **10**, 283, 514.] *D. H. Lehmer.*

Ward, Morgan. Arithmetical properties of the elliptic polynomials arising from the real multiplication of the Jacobi functions. Amer. J. Math. **72**, 284–302 (1950).

There are four sets of polynomials A_n, B_n, C_n, D_n associated with the real multiplication of Jacobi's functions sn u, cn u, dn u. They are polynomials in $\text{sn}^2 u$, k^2 with rational integral coefficients and, for example,

$$\text{sn } nu / \text{sn } u = B_n(\text{sn}^2 u, k^2) / A_n(\text{sn}^2 u, k^2)$$

if n is odd,

$$\text{sn } nu / \text{sn } u = \text{cn } u \text{ dn } u B_n(\text{sn}^2 u, k^2) / A_n(\text{sn}^2 u, k^2)$$

if n is even. If $\text{sn}^2 u = x$ and $k^2 = a$ where x and a are fixed algebraic integers the polynomials form four sequences of algebraic integers. The arithmetical properties of these sequences are studied. The results and methods are similar to those for the polynomials associated with the real multiplication of the $\wp$ function [Amer. J. Math. **70**, 31–74 (1948); Duke Math. J. **15**, 941–946 (1948); these Rev. **9**, 332; **10**, 283]. Not all the sequences are elliptic divisibility sequences but the laws of repetition and apparition for powers of primes are obtained. Certain particular cases either reduce to or generalize the laws of repetition and apparition for Lucas' series. *H. S. Zuckerman.*

Mardžanišvili, K. K. On a system of equations in prime numbers. Doklady Akad. Nauk SSSR (N.S.) **70**, 381–383 (1950). (Russian)

Suppose $l < m < \cdots < n$ are fixed positive integers, g in number, and let $N_l < N_m < \cdots < N_n$ be variable positive integers. For suitable s the author considers the system of equations

$$(*) \quad p_1^l + \cdots + p_s^l = N_l,\ p_1^m + \cdots + p_s^m = N_m,\ \cdots,\ p_1^n + \cdots + p_s^n = N_n$$

Put $N_n = P^n$, $N_k = h_k P^k$ $(k = l, m, \cdots, n)$, and

$$\Delta(z_1, \cdots, z_g) = \begin{vmatrix} z_1^{l-1} & \cdots & z_g^{l-1} \\ \cdots & \cdots & \cdots \\ z_1^{n-1} & \cdots & z_g^{n-1} \end{vmatrix}.$$

The author states the following theorem. If f is a whole number not less than $3ng$, if $s = f + 2gr$, where $r = [2n \log 10ng + n \log \log 20ng + 1]$, and if the system of equations $\xi_1^k + \cdots + \xi_f^k = h_k$ $(k = l, m, \cdots, n)$ is solvable in real numbers $\xi_1, \cdots, \xi_f$ satisfying the conditions $\xi_j \geqq \epsilon$ $(j = 1, 2, \cdots, f)$ and $|\Delta(\xi_1, \cdots, \xi_g)| \geqq \epsilon$, where ϵ is some (fixed) positive number, then the number of solutions, $I = I(N_l, \cdots, N_n; l, \cdots, n; s)$, of the system (*) satisfies the inequality

$$I \geqq P^{f+2gn\left(1 - \frac{(1-1/n)\sum_{l \leq i \leq n} i}{\sum_{l \leq i \leq n} (i - 1/n)}\right)} \{\log P\}^{-f-2gr} \{CS + O(\log P)^{-\omega}\},$$

where S is the singular series and C, ω, and the O-constant are positive numbers depending only on $l, m, \cdots, n, f, g$, and ϵ. The author also states two theorems giving conditions under which the singular series is bounded away from zero.

In earlier publications [Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] **4**, 193–214 (1940); Uspehi Matem. Nauk (N.S.) **4**, no. 1(29), 183–185 (1949); these Rev. **2**, 250; **11**, 161] the author has considered the special case in which $l, m, \cdots, n$ are the integers $1, 2, \cdots, n$ (the so-called simultaneous Waring-Goldbach problem). For a general survey of the author's work see the résumé of his doctoral dissertation [see the following review]. *P. T. Bateman* (Urbana, Ill.).

Mardžanišvili, K. K. Investigations on the application of the method of trigonometric sums to additive problems. Uspehi Matem. Nauk (N.S.) **5**, no. 1(35), 236–240 (1950). (Russian)

This paper is a résumé of the author's doctoral dissertation. Besides the material of the dissertation itself, it includes a brief history of additive number theory, with specific reference to the Waring and Goldbach problems and their generalizations. In the first two chapters of the dissertation the author treats the system of Diophantine equations $x_1^l + \cdots + x_s^l = N_l$, $x_1^m + \cdots + x_s^m = N_m$, $\cdots$, $x_1^n + \cdots + x_s^n = N_n$, where $l < m < \cdots < n$ are fixed positive integers, g in number. He proves that this system is solvable for an s having the order of magnitude $ng \log n$, provided $N_l, \cdots, N_n$ satisfy appropriate order conditions and certain arithmetical requirements. In chapter 3 the author gives a careful estimation of the error term in the Vinogradov formula for the number of representations of a large odd number as a sum of three primes, with a view toward getting a good value for the "Vinogradov constant." In

chapter 4 he proves that the asymptotic formula in the simultaneous Waring-Goldbach problem is valid for a number of summands of the order $n^2 \log n$ [see the second paragraph of the preceding review and also Hua, Quart. J. Math., Oxford Ser. 20, 48–61 (1949); these Rev. 10, 597]. In chapter 5 he discusses the singular series for the system (*) of equations in prime numbers considered in the paper reviewed above, and in chapter 6 he proves the asymptotic inequality for the number of solutions which is stated in the preceding review. Finally in chapter 7 he derives the asymptotic formula and studies the singular series for the system $p_1+\cdots+p_s=N_1$, $p_1^2+\cdots+p_s^2=N_2$, with $s\geqq 7$ (and $p_1, \cdots, p_s$ prime numbers). *P. T. Bateman.*

van der Corput, J. G., and Kemperman, J. H. B. The second pearl of the theory of numbers. I. Nederl. Akad. Wetensch., Proc. 52, 696–704 = Indagationes Math. 11, 226–234 (1949).

van der Corput, J. G., and Kemperman, J. H. B. The second pearl of the theory of numbers. II. Nederl. Akad. Wetensch., Proc. 52, 801–809 = Indagationes Math. 11, 277–285 (1949).

van der Corput, J. G., and Kemperman, J. H. B. The second pearl of the theory of numbers. III. Nederl. Akad. Wetensch., Proc. 52, 927–937 = Indagationes Math. 11, 325–335 (1949).

The authors extend the (α, β)-theorem to abstract sets. Their main result, theorem 7, is too complicated to be quoted here, but their theorem 4 will indicate the direction of their generalizations. Given a transitively ordered set G with a smallest element 0. In G a commutative and associative addition is defined such that (1) $g+0=g$, (2) $g+g'=g+g''$ implies $g'=g''$, (3) $g>0$ implies $g+g'>g'$. Let $\varphi(g)$ be a mapping of the positive elements of G into the real axis such that $g\leqq g'+g''$ implies $\varphi(g)\leqq\varphi(g')+\varphi(g'')$ $(g, g', g''\subset G)$. Let A and B be finite subsets of a subset H of G; $0\subset A$, $0\subset B$; and let $A+B$ denote the set of all the elements $a+b$ $(a\subset A, b\subset B)$. Suppose finally that $A(h)+B(h)\geqq 1+\varphi(h)$ for every $h\subset H$, $h>0$. Then $(A+B)(h)\geqq\varphi(h)$ for these h's. Here $A(h)$ denotes the number of $a\subset A$ with $a<h$, etc.

In part I, theorem 4 is proved and various applications and special cases are given. Part II deals with generalizations of this theorem and their applications. All of them are contained in or follow from theorem 7 which is proved in part III. The main features of theorem 7 are the following. (1) The assumptions on G are weakened. (2) A whole set of functions $\varphi(g)$ is considered. (3) To each $\varphi(g)$ a weight function $f(g)$ is associated and the above definition of $A(h)$ is replaced by $A(h)=\sum f(a)$, summed over $a\subset A$, $a<h$. *P. Scherk* (Saskatoon, Sask.).

Hasse, Helmut. Arithmetische Bestimmung von Grundeinheit und Klassenzahl in zyklischen kubischen und biquadratischen Zahlkörpern. Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. 1948, no. 2, 95 pp. (1950).

The author shows how the analytic theory of algebraic number fields, involving Gaussian sums and their relation to class numbers, may be exploited for the establishment of systematic methods and algorithms for the determination of unit groups and class numbers. Naturally the intricacies of the analytic relations become extremely unmanageable if they are applied to specific fields. Thus, the author limits himself to the study of cyclic cubic and biquadratic extensions K/P of the rational number field. Theoretically the adequate tool is the imbedding of K/P into the extension KP_ν, where P_ν denotes the field of all νth roots of unity; here $\nu=3, 4$. Then the theory of Gaussian sums (belonging to the character corresponding to K/P as a class field) determines in an invariantive fashion a generating radical of KP_ν/P_ν. The laws of decomposition of K/P imply further the arithmetical determination of the respective radicands. Next the author has to set up bases of the integers in the fields K, which are more readily manageable than the ones known from the customary processes of normalization which are based on congruence theory. Here the integral elements of K are characterized as invariant integral elements of KP_ν, and many useful formulas for explicit computations are obtained. By means of these formulas, depending on the field K, various classifications of the units of K are given and inequalities for the coefficients of the units are obtained, and the author finds minimal properties which characterize the respective fundamental units. A combination of these results also leads to algorithms for the determination of the index of a unit in another. Such results are important for the evaluation of class numbers since the analytic description of the latter involves the index of the system of cyclotomic units in the system of all units for the maximal real subfield of K. The tool for finding explicit formulas is furnished by Bergstrom's formula [J. Reine Angew. Math. 186, 91–115 (1944); these Rev. 7, 148]. Finally many tables are appended to this paper, which are very useful for the unavoidable work on explicit examples which is necessary in number theory. *O. F. G. Schilling.*

ANALYSIS

Jecklin, Heinrich. La notion de moyenne. Metron 15, 3–11 (1949).

The author summarizes a number of Italian mathematical and statistical papers and some of his own publications on the descriptive theory of internal means, i.e., those such that $\min(x_1, \cdots, x_n)\leqq m(x_1, \cdots, x_n)\leqq\max(x_1, \cdots, x_n)$. *J. Aczél* (Miskolc).

Mulholland, H. P. On generalizations of Minkowski's inequality in the form of a triangle inequality. Proc. London Math. Soc. (2) 51, 294–307 (1950).

For values $a_j\geqq 0$, $b_j\geqq 0$, $q_j>0$, $\sum q_j=1$; $j=1, \cdots, n$, and for a continuous and strictly monotonic function $\varphi(x)$, sums and means are defined respectively by

$$\mathfrak{S}_\varphi(a)=\varphi^{-1}[\sum\varphi(a)],\quad \mathfrak{M}_\varphi(a)=\varphi^{-1}[\sum q\varphi(a)].$$

The author studies the inequalities

(SS) $\mathfrak{S}_\varphi(a+b)\leqq\mathfrak{S}_\varphi(a)+\mathfrak{S}_\varphi(b)$,

(MS) $\mathfrak{M}_\varphi(a+b)\leqq\mathfrak{M}_\varphi(a)+\mathfrak{M}_\varphi(b)$,

and the corresponding inequalities with sign of inequality reversed, and also includes for completeness known results concerning

(SM) $\mathfrak{S}_\varphi(\frac{1}{2}a+\frac{1}{2}b)\leqq\frac{1}{2}\mathfrak{S}_\varphi(a)+\frac{1}{2}\mathfrak{S}_\varphi(b)$,

(MM) $\mathfrak{M}_\varphi(\frac{1}{2}a+\frac{1}{2}b)\leqq\frac{1}{2}\mathfrak{M}_\varphi(a)+\frac{1}{2}\mathfrak{M}_\varphi(b)$,

and their reverse inequalities. It is shown that the inequal-

ity (AB), where A and B independently represent either S or M, holds provided that, for $x \geqq 0$, $\varphi(x)$ is continuous, strictly increasing, and convex, with $\varphi(0)=0$, and the function $F_{AB}(t)$, defined by

$$\begin{aligned} F_{SS}(t) &\equiv \log \varphi(e^t), && -\infty < t < \infty, \\ F_{MS}(t) &\equiv -\log |\varphi'(e^t)|, && -\infty < t < \infty, \\ F_{SM}(t) &\equiv \varphi(t)/\varphi'(t), && 0 < t < \infty, \\ F_{MM}(t) &\equiv -\varphi'(t)/\varphi''(t), && 0 < t < \infty, \end{aligned}$$

is convex throughout its range (where a derivative of $\varphi(x)$ appears in these formulas, it is supposed to be continuous). Conditions implying the reverse inequalities also are given in terms of the same functions F. In addition, relations between the inequalities, and also necessary conditions, are discussed. *E. F. Beckenbach* (Los Angeles, Calif.).

Taylor, A. E. A note on an inequality for integrals. Amer. Math. Monthly **57**, 93–96 (1950).

This note shows that for a Banach-space-valued continuous function on $a \leqq x \leqq b$, the equality

$$\left\| \int_a^b f(x)dx \right\| = \int_a^b \|f(x)\| dx$$

holds if and only if, for every finite set of points x_i on $[a, b]$,

$$\|\textstyle\sum_i f(x_i)\| = \sum_i \|f(x_i)\|.$$

(Geometrically this means that all values $f(x)$, $a \leqq x \leqq b$, lie in the cone of half rays from the origin through a convex subset of the surface of the unit sphere.) Hence for B strictly convex, the first equality holds if and only if $f(x)=g(x)b$, where $b \in B$, $b \neq 0$, and $g(x) \geqq 0$ for all x.

M. M. Day (Urbana, Ill.).

Rodov, A. Relations between the upper bounds of the derivatives of a function of a real variable. Amer. Math. Soc. Translation no. **14**, 32 pp. (1950).

Translated from Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 257–270 (1946); these Rev. **8**, 65.

Stečkin, S. B. On bilinear forms. Doklady Akad. Nauk SSSR (N.S.) **71**, 237–240 (1950). (Russian)

Theorem 1. If c_k are the Fourier coefficients on $(0, 1)$ of a function $f(t)$ of L^r, $r>1$, and if $1<p \leqq 2$, $1<q \leqq 2$, $p'^{-1}+q'^{-1}+r^{-1}=1$, the form $T=\sum\sum c_{m-n}x_m y_n$ is bounded in space $[p, q]$ with bound not exceeding $\{\int |f|^r dt\}^{-1}$. The case $r=\infty$, $p=q=2$, is due to Toeplitz [see Hardy, Littlewood and Pólya, Inequalities, Cambridge University Press, 1934, theorem 303]. Theorem 2. If $f(t)$ is of bounded variation, T is bounded in $[p, p']$ for $p>1$. The theorems are also rephrased in terms of operations on l^p spaces.

R. P. Boas, Jr. (Providence, R. I.).

Theory of Sets, Theory of Functions of Real Variables

***Halmos, Paul R. Measure Theory.** D. Van Nostrand Company, Inc., New York, N. Y., 1950. xi+304 pp. $5.90.

This book presents a unified theory of the general theory of measure and is intended to serve both as a text book for students and as a reference book for advanced mathematicians. This book is written in a very clear style and will make an excellent text book for those graduate students who are already familiar with the theory of Lebesgue integration in a Euclidean space. But the treatment of the subject is rather abstract so that the book is perhaps not to be recommended for beginners. On the other hand, this book contains many new results obtained in the last ten years, and will be a very useful source of reference for research mathematicians. Those people who are familiar with the classical theory of measure and integration will find many improvements and simplifications in various parts of this book. The bibliography at the end of the book is extensive and up-to-date, but is not claimed to be exhaustive. The book does not cover all aspects of measure theory. For example, Vitali's covering theorem and density theorems are not treated.

The book consists of a preparatory section [§ 0] and 12 chapters [§§ 1–64]. At the end of each section there is a set of problems, with occasional hints for solution. Some of these problems are important results which would better be stated as theorems in the main body of the book. For example, problem 40–12 is a result well known as the Vitali-Hahn-Saks theorem and is one of the most useful results in the theory of Lebesgue integration. Among these problems there are also many results which are published here for the first time. Two examples due to Dieudonné given as problems 48–4 and 49–3 deserve attention, too.

In the first three chapters the author discusses fundamental concepts concerning families of sets and countably additive measures defined on them. As the domain of measure σ-rings are taken instead of the more customary σ-fields. (A family $\mathbf{S}$ of subsets E of a space X is a σ-ring if (i) $E, F \in \mathbf{S}$ imply $E-E \cap F \in \mathbf{S}$, (ii) $E_i \in \mathbf{S}$, $i=1, 2, \cdots$ imply $\bigcup_{i=1}^{\infty} E_i \in \mathbf{S}$. The family $\mathbf{S}$ is a σ-field if it further satisfies (iii) $E \in \mathbf{S}$ implies $X-E \in \mathbf{S}$.) It is not required that the whole space is the union of a countable number of subsets of finite measure. This general approach enables the author to give a unified treatment of measure theory when measures on locally compact (but not necessarily compact nor σ-compact) topological spaces and topological groups are discussed in chapters X, XI and XII.

Chapters IV and V are devoted to the theory of measurable functions and Lebesgue integration. Given a σ-ring $\mathbf{S}$ of subsets of a space X, a real-valued function $f(x)$ defined on X is called $\mathbf{S}$-measurable if the sets $\{x | f(x)>a\}$ and $\{x | f(x)<b\}$ belong to $\mathbf{S}$ for any $a>0$ and for any $b<0$, while the set $\{x | f(x)=0\}$ does not need to belong to $\mathbf{S}$. This modification of the usual definition is necessary since $\mathbf{S}$ is not necessarily a σ-field. After discussing various kinds of convergence with respect to a countably additive measure $\mu(E)$ defined on $\mathbf{S}$ (almost everywhere convergence, almost uniform convergence and convergence in measure) the Lebesgue integral is defined as follows: first the integral of a simple function $f(x)$ (i.e., a function of the form: $f(x)=\sum_{i=1}^{k} a_i \chi_{E_i}(x)$, where $\chi_E(x)$ is the characteristic function of a measurable set $E \in \mathbf{S}$ with a finite μ-measure) is defined by $\int f(x)d\mu(x)=\sum_{i=1}^{k} a_i \mu(E_i)$. Then an $\mathbf{S}$-measurable function $f(x)$ is called integrable if it is a limit in measure of a sequence of simple functions $\{f_n(x)\}$ such that

$$\lim_{m,n \to \infty} \int |f_m(x)-f_n(x)| d\mu(x) = 0,$$

and the integral of $f(x)$ is defined by

$$\int f(x)d\mu(x) = \lim_{n \to \infty} \int f_n(x)d\mu(x).$$

This definition of the Lebesgue integral, which is due to

Dunford [Trans. Amer. Math. Soc. **37**, 441–453 (1935); **38**, 600–601 (1935)] has the advantage of avoiding all complicated arguments in the definition of integrability and the integral of unbounded functions. Further, various properties of the Lebesgue integral (e.g., additivity of the integral with respect to integrands and absolute integrability of unbounded integrable functions) follow immediately from the definition.

In chapter VI general properties of signed measures (i.e., real-valued countably additive set functions which are not necessarily nonnegative and which can take one, but not both, of the values $\pm\infty$) are discussed. The Hahn-Jordan decomposition (into positive and negative parts) and the Lebesgue decomposition (into absolutely continuous and singular parts) are obtained. A very simple proof of the Radon-Nikodým theorem due to Yosida is given. Chapter VII is on product measures. First, direct product measures on finite direct product spaces are discussed in a very clear way. The author's treatment of Fubini's theorem is simpler than anything previously published. Infinite direct product measures are then discussed following the method of Andersen-Jessen and the reviewer.

Various problems concerning measure spaces (a measure space is a triple $(X, \mathbf{S}, \mu)$ of a space X, a σ-ring $\mathbf{S}$ of subsets E of X, and a countably additive measure $\mu(E)$ defined on $\mathbf{S}$) are discussed in chapter VIII: measurability and measure preserving transformations, measure rings and measure algebras, isomorphism and representation theorems, functional spaces (L^p-spaces, $1\leqq p<\infty$) and the inequalities of Hölder and Minkowski. Relations between set functions and point functions are discussed in the case when the space is the real line. Application of measure theory to probability is the subject of chapter IX. Following Kolmogoroff it is shown that the theory of probability is equivalent to the theory of measure spaces $(X, \mathbf{S}, \mu)$ with $\mu(X)=1$. The notion of independence (in the sense of probability) is introduced. The inequality of Kolmogoroff and the Khintchine-Kolmogoroff three series theorem are proved. The weak and strong laws of large numbers are discussed. Conditional probabilities and nondirect product measures on direct product spaces are discussed.

Properties of measures on a locally compact (but not necessarily compact nor σ-compact) Hausdorff space X are treated in chapter X. Borel sets (Baire sets) of X are defined as sets belonging to the σ-ring $\mathbf{S}$ ($\mathbf{S}_0$) generated by the family of all compact sets (compact G_δ-sets) of X. Countably additive measures defined on $\mathbf{S}$ ($\mathbf{S}_0$) are called Borel measures (Baire measures). Regularity of Borel and Baire measures is discussed. This chapter concludes with a proof of the theorem that every positive bounded linear functional defined on the space of all those real-valued continuous functions on X which vanish outside a compact subset of X can be represented as an integral with respect to a regular Borel measure on X.

Chapters XI and XII are devoted to the theory of Haar measure. In chapter XI the existence of a left invariant (right invariant) Haar measure on a locally compact topological group (without any countability restrictions) is proved by using Haar's original idea. Tychonoff's theorem is used instead of Cantor's diagonal process or Banach limits. The uniqueness of Haar measure is then proved by using Fubini's theorem. In chapter XII the author first discusses the results of Weil and Kodaira concerning the relations between measure and topology on a group. Further, the results of Ambrose concerning the relations between the Haar measure of a locally compact group X and that of the factor group X/Y, where Y is a compact invariant subgroup of X, are discussed. Finally, following Kodaira and the reviewer, this result is applied to prove that every Haar measure on a locally compact topological group is completion regular, i.e., that the Haar-Borel measure and the Haar-Baire measure (=the restriction of the Haar-Borel measure to Baire sets) have the same completion.

S. Kakutani (New Haven, Conn.).

Berger, Agnes. On disjoint sets of distribution functions. Proc. Amer. Math. Soc. **1**, 25–31 (1950).

Let I, J be two classes of one-dimensional distributions P_i, Q_j such that for no i, j is $P_i=Q_j$. Call the pair I, J distinguishable if there is a Borel set W such that for no i, j is $P_i(W)=Q_j(W)$. If $I=\{1\}$, $J=\{1, 2\}$, I, J are always distinguishable, but for any P with at least two jumps, there always exist Q_1, Q_2, Q_3 such that $\{P\}$, $\{Q_1, Q_2, Q_3\}$ are not distinguishable. If $I=\{1, 2\}$, $J=\{1, 2\}$, I, J are not always distinguishable. However, if all P_i, Q_j are continuous, I, J are distinguishable if each is denumerable.

D. Blackwell (Washington, D. C.).

Cinquini, Silvio. Sopra il cambiamento delle variabili negli integrali doppi. Boll. Un. Mat. Ital. (3) **4**, 228–235 (1949).

By factoring a biunique plane transformation of class C^1 into two plane transformations, the author establishes the ordinary formula for changing variables in a double integral and in doing so he uses only a special case of Green's theorem involving a region defined by inequalities of the form $a\leqq x\leqq b$, $\phi(x)\leqq y\leqq\psi(x)$ or $c\leqq y\leqq d$, $\phi(y)\leqq x\leqq\psi(y)$.

R. G. Helsel (Columbus, Ohio).

Stewart, Frank M. Integration in noncommutative systems. Trans. Amer. Math. Soc. **68**, 76–104 (1950).

In this paper the author introduces an abstract process of integration of "differentials." His theory includes not only the known additive integral of functions with values in a Banach space, but also the multiplicative integral, whose study was begun by V. Volterra at the end of the last century in connection with his work on differential equations [more recent contributions to the multiplicative integral theory are due to L. Schlesinger, Math. Z. **33**, 33–61 (1931); **35**, 485–501 (1932); G. Rasch, J. Reine Angew. Math. **171**, 65–119 (1934); G. Birkhoff, J. Math. Physics **16**, 104–132 (1937); P. R. Masani, same Trans. **61**, 147–192 (1947); these Rev. **8**, 321]. The differentials are certain functions $\mu\{t, \sigma\}$ of a real variable $t\varepsilon I=[\alpha, \beta]$ and of a Lebesgue measurable set $\sigma\subset I$. The values of μ are in a complete metric space $\mathfrak{D}$, with distance ρ, provided with a continuous associative operation $\circ:\mathfrak{D}\times\mathfrak{D}\to\mathfrak{D}$ which has an identity e. The following condition of "relative stability," which replaces convexity in the additive case, plays an important role in the discussion: given $L>0$, $\epsilon>0$ there is $\delta>0$ such that $\sum_1^n\rho(x_i, e)\leqq L$, $\sum_1^n\rho(x_i, y_i)\leqq\delta$ imply $\rho(x_1\circ\cdots\circ x_n, y_1\circ\cdots\circ y_n)<\epsilon$. Most of the paper discusses the existence and elementary properties of integrals $\int_\alpha^\beta\mu\{t, \sigma\}$ of the Riemann and Lebesgue type (defined as suitable limits of expressions of the type $\mu\{t_1, \sigma_1\}\circ\cdots\circ\mu\{t_n, \sigma_n\}$). These integrals subsume the additive and multiplicative integrals through the following differentials: if $\mathfrak{D}$ is a Banach space and the operation $\circ$ is addition, every function $f:I\to\mathfrak{D}$ determines the differential $f(t)|\sigma|$, where $|\sigma|$ stands for the Lebesgue measure of σ; and if $\mathfrak{D}$ is a Banach algebra with

unity and o is multiplication, every $f: I \to \mathfrak{D}$ determines the differential $1+f(t)|\sigma|$. *L. Nachbin* (Chicago, Ill.).

Simonsen, W. Sur les correspondances multivoques entre deux ensembles abstraits. Acta Math. **81**, 291–297 (1949).

If f is a correspondence between two sets A and B, $S \subset A$ is called invariant if $f^{-1}fS = S$. Various simple properties of the class of invariant sets are studied. *R. Arens.*

Šnelder, V. E. Descriptive theory of sets in topological spaces. Učenye Zapiski Moskov. Gos. Univ. **135**, Matematika, Tom II, 37–85 (1948). (Russian)

This memoir extends a number of classical results concerning Borel sets and analytic sets from the case of subsets of a separable complete metric space to the case of subsets of more general topological spaces. The following conditions on the spaces X discussed are frequently employed: (A) every closed subset of X is a G_δ; (B) if $\{G_\alpha\}$, $\alpha \in A$, is any family of open subsets of X, then there exists a countable subfamily $\{G_{\alpha_n}\}_{n=1}^{\infty}$ of $\{G_\alpha\}$, $\alpha \in A$, such that $\bigcup_{\alpha \in A} G_\alpha = \bigcup_{n=1}^{\infty} G_{\alpha_n}$. (The author describes spaces satisfying condition (B) as spaces of countable structure.) The Borel families constructed in the usual way by the operations of countable intersections and countable unions, starting with the families of open sets and closed sets, may be distinct, in general spaces, as may the corresponding families of analytic sets. A set analytic with respect to the family of closed sets is called an F_A-set. The distinctions just described vanish, of course, for spaces satisfying condition (A).

A number of simple topological theorems are first proved, among them the facts that conditions (A) and (B) are equivalent for compact (bicompact in the author's terminology) Hausdorff spaces and that every regular space satisfying condition (B) is completely normal. Next, the following space L is studied in detail. The space L is made up of two intervals $I = E[x; 0 \leqq x < 1]$ and $I' = E[x'; 0 < x' \leqq 1]$. A generic neighborhood of $x_0 \in I$ is $[x_0, x_0+t) \cup (x_0', x_0'+t)$; a generic neighborhood of $x_0' \in I'$ is $(x_0'-t, x_0'] \cup (x_0-t, x_0)$, where $x_0 = x_0'$ and t is an arbitrary positive real number. It is proved that L is a compact Hausdorff space satisfying condition (B) which is nonmetrizable and that no uncountable analytic subset of L can be represented as a continuous image of a subspace of the space of irrational numbers.

A large number of theorems concerning analytic and Borel sets are proved, of which the following may be taken as typical. (1) In a compact Hausdorff space satisfying condition (B), every pair of disjoint analytic sets are contained in a pair of disjoint Borel sets. (2) In a compact Hausdorff space satisfying condition (B), an analytic set whose complement is analytic is a Borel set. (3) In a compact Hausdorff space satisfying condition (B), an analytic set is a Borel set if and only if it is representable by means of disjoint summands. (4) Every zero-dimensional compact Hausdorff space which is dense in itself and satisfies condition (B) contains an analytic set which is not a Borel set. (5) If X is a completely regular space such that X is an F_A-set in βX, then X is an F_A-set in every compact Hausdorff space containing X as a dense subspace. (6) Let f be a continuous mapping carrying a space X satisfying the hypotheses of (5) into a compact Hausdorff space Y. Then $f(X)$ is an F_A-set in Y. Finally, generalized notions of Borel set and analytic set are introduced, the ordinary countable operations being replaced by unions and intersections of larger families of sets and a corresponding generalization being presented for the operation (A). Some of the results obtained for ordinary Borel and analytic sets are extended to these more general cases. *E. Hewitt* (Seattle, Wash.).

Šilov, G. E. Generalization of a theorem on the differentiation of a uniformly convergent sequence of functions. Mat. Sbornik N.S. **26**(68), 75–84 (1950). (Russian)

Call a function α defined on the interval $a \leqq t \leqq b$ distinguished if and only if (1) $0 \leqq \alpha(t) \leqq 1$ for $a \leqq t \leqq b$, and (2) for every sequence $\{x_n\}$ with continuous derivatives $\{x_n'\}$ on $a \leqq t \leqq b$, such that (a) $\{x_n\}$ converges uniformly on $a \leqq t \leqq b$ to zero, and (b) $\{\alpha x_n'\}$ converges uniformly on $a \leqq t \leqq b$ to a function ψ, it follows that ψ is identically zero. A well-known theorem of analysis asserts that the constant functions on $a \leqq t \leqq b$ are distinguished. Lemma. Suppose $0 \leqq \alpha(t) \leqq 1$ on $a \leqq t \leqq b$ and let $\beta(t) = \max[\alpha(t), \limsup_{s \to t} \alpha(s)]$; then α is distinguished if and only if the upper semi-continuous function β is distinguished.

For each positive constant C let F_C be the set of those points t such that $\{s \mid \alpha(s) \geqq C\}$ is of density one at t and $1/\alpha$ is summable in an interval around t; let $Q_1 = \bigcup_{C>0} F_C$. If Q_λ is defined for an ordinal λ, let $Q_{\lambda+1} = \bigcup_{C>0}$ [closure of $Q_\lambda \cap F_C$]; if, for a limit ordinal λ, Q_μ is defined for $\lambda > \mu$, let $Q_\lambda = \bigcup_{\mu<\lambda} Q_\mu$. Theorem. An upper semi-continuous α, $0 \leqq \alpha(t) \leqq 1$, is distinguished if and only if each point t where $\alpha(t) > 0$ is in at least one of the transfinite sequence of sets Q_λ. A second characterization is as follows. Let $S = \{t \mid \alpha(t) > 0\}$ and let $S_n = \{t \mid \alpha(t) \geqq 1/n\}$. Define a topology in S by taking as a basis of closed sets the zero sets of functions x defined on S and continuous on each S_n. If α is upper semi-continuous, α is distinguished if and only if Q_1 is dense in this new topology in S. *M. M. Day.*

Viola, Tullio. Sul modo d'approssimare una curva rettificabile, descritta su una superficie quadrabile, mediante altre curve rettificabili. Rivista Mat. Univ. Parma **1**, 45–57 (1950).

On a surface S, let there be given a curve C with end points A, B. Let Λ be the smallest number that can be obtained as the limit of the lengths $L(C_n)$ of curves C_n satisfying the following conditions. (i) The end points of C_n coincide with A, B; (ii) C_n has no points in common with S, except for the end points A, B; (iii) $C_n \to C$. Let Λ' be defined in a similar manner, except that (ii) is replaced by the condition that C_n lies on S. The quantities $\Lambda, \Lambda', L(C)$ then satisfy the inequalities $\Lambda \geqq L(C)$, $\Lambda' \geqq L(C)$. The author constructs examples showing that each one of the following combinations is possible:

(1) $\Lambda = L(C)$, $\Lambda' = L(C)$; (2) $\Lambda > L(C)$, $\Lambda' > L(C)$;
(3) $\Lambda > L(C)$, $\Lambda' = L(C)$; (4) $\Lambda = L(C)$, $\Lambda' > L(C)$.

T. Radó (Columbus, Ohio).

Torrigiani, Guido. Sulle funzioni di più variabili a variazione limitata. Rivista Mat. Univ. Parma **1**, 59–83 (1950).

It was shown by Saks [Fund. Math. **22**, 257–261 (1934)] that there is a function $\phi(x, y)$, nonnegative and summable on the square $Q[0, 1; 0, 1]$, such that when $(x_0, y_0) \in Q$,

$$\limsup_{h, k \to 0} (4hk)^{-1} \int_{x_0-h}^{x_0+h} \int_{y_0-k}^{y_0+k} \phi(x, y)\,dx\,dy = \infty,$$

$\phi(x, y)$ being taken as 0 outside Q. Such a function will be said to be of class S. The author shows that total variation functions derived from a continuous function of three vari-

ables may belong to S. More precisely, let $f(x, y, z)$ be defined on the closed cube $C[0, 1; 0, 1; 0, 1]$, let $V_x(x_0, y_0, z_0)$ be its total variation for $0 \leqq x \leqq x_0$, $y = y_0$, $z = z_0$, and write $V_x(y, z)$ for $V_x(1, y, z)$: similarly for V_y and V_z. Then $f(x, y, z)$ is said to be of bounded variation in Tonelli's sense (BVT) when $V_x(y, z)$, $V_y(z, x)$, $V_z(x, y)$ are all summable on Q. Theorem. There is a function $f(x, y, z)$, continuous and BVT on C and vanishing on its boundary, such that $V_x(y, z)$, $V_y(z, x)$, $V_z(x, y)$ are everywhere finite, $V_x(y, z)$ and $V_y(z, x)$ are bounded, and $V_z(x, y)$ and also all the differences $V_z(x, y, z_0+\lambda) - V_z(x, y, z_0)$ $[0 \leqq z_0 < z_0+\lambda \leqq 1]$ belong to S. The author's constructions are based partly on that employed by Saks. His results show that it is not possible to extend to functions of three variables a proof by Tonelli [Ann. Scuola Norm. Super. Pisa (2) 5, 315–320 (1936)] of a theorem due to Cesari on convergence of double Fourier series. *H. P. Mulholland* (Beirut).

Theory of Functions of Complex Variables

Taylor, Angus E. New proofs of some theorems of Hardy by Banach space methods. Math. Mag. 23, 115–124 (1950).

According to classical theorems, for any complex valued function $f(z)$ holomorphic in the unit circle, the maximum value function $M(f; r) \equiv \max |f(re^{i\theta})|$, $0 \leqq \theta < 2\pi$, is an increasing function of r, and $\log M(f; r)$ is a convex function of $\log r$; the latter result, due to Hadamard, is usually called the three circles theorem. G. H. Hardy [Proc. London Math. Soc. (2) 14, 269–277 (1915)] generalized these results in showing that they still hold when $M(f; r)$ is replaced by the mean value function

$$\mathfrak{M}_p(f; r) \equiv \left\{ (2\pi)^{-1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}$$

for any positive p. The author now extends these results to holomorphic functions of a complex variable, with function values in an arbitrary Banach space $\mathfrak{B}$. Further, by suitably specializing $\mathfrak{B}$, with ordinary holomorphic functions $f(z)$ as elements, he associates with each $f(z)$ a holomorphic function $\mathfrak{F}(z)$ with values in $\mathfrak{B}$ such that the maximum of the norm of $\mathfrak{F}(z)$ on $|z| = r$ is $\mathfrak{M}_p(f; r)$. As thus treated, Hardy's results appear as special instances of generalizations of the classical theorems to functions with values in Banach spaces, rather than as extensions of the classical theorems to mean value functions. The method as it stands, however, requires the limitation $p \geqq 1$.

E. F. Beckenbach (Los Angeles, Calif.).

Aumann, Georg. Über die Streckenverzerrung bei konvexen konformen Abbildungen. S.-B. Math.-Nat. Kl. Bayer. Akad Wiss. 1948, 303–308 (1949).

Let $w = f(s)$ map the unit circle $|s| < 1$ conformally onto a convex region $\mathfrak{F}$ with $w_1 = f(s_1)$, $w_2 = f(s_2)$, where $|s_1| < 1$, $|s_2| < 1$, and $s_1 \neq s_2$. Then let $w_3 = h_1 w_1 + h_2 w_2$ for $h_1 > 0$, $h_2 > 0$, and $h_1 + h_2 = 1$. Let $s = g(w)$ represent the inverse of $w = f(s)$. The author proves that the region of variability for $g(w_3)$, as f varies through the class of all convex mappings of $|s| < 1$, with s_1, s_2, h_1, and h_2 fixed, is precisely a circle in $|s| < 1$. *G. Springer* (Cambridge, Mass.).

Andreotti, Aldo. Un'applicazione di un teorema di Cecioni ad un problema di rappresentazione conforme. Ann. Scuola Norm. Super. Pisa (3) 2 (1948), 99–103 (1950).

Le théorème de Cecioni [Ann. Univ. Toscane (N.S.) 12, 27–88 (1929)] utilisé par l'auteur étend aux domaines appartenant à une surface de Riemann à un nombre fini de feuillets le théorème classique de Schottky relatif à la représentation conforme biunivoque des domaines plans multiplement connexes: il permet de représenter un tel domaine sur un autre domaine convenable appartenant à une surface de Riemann algébrique orthosymétrique. L'auteur montre qu'un domaine D de cette dernière espèce limité par $\rho+1$ contours composés d'arcs analytiques est représentable conformément et biunivoquement sur un domaine analogue dont les $\rho+1$ contours sont des circonférences simples ou multiples de la surface de Riemann dont les centres se projettent au même point du plan simple. Il signale que le cas $\rho = 0$ avait déjà été traité par Matildi [voir mêmes Ann. (2) 14 (1945), 81–90 (1948); ces Rev. 10, 241] et observe que le résultat s'étend au cas où tous les points des contours sont accessibles de l'intérieur de D, donc en particulier si ce sont des courbes de Jordan. La démonstration de nature algébrico-transcendante est basée sur l'emploi des intégrales abéliennes.

G. Valiron (Paris).

Walsh, J. L. On distortion at the boundary of a conformal map. Proc. Nat. Acad. Sci. U. S. A. 36, 152–156 (1950).

Let the function $w = f(z)$ map the half-plane R_z $(x < 1; z = x+iy)$ onto a bounded Jordan region R_w, containing the line segment $0 \leqq w < 1$, whose boundary possesses at the point $w = 1$ forward and backward tangents making equal angles $\alpha/2$ (>0) with the negative direction of the real axis. Let $f(0) = 0$, $f(1) = 1$. Consider a sequence x_n $(0 < x_n < 1)$, where $x_n \to 1$, and the functions

$$f_n(z) = \{f[(1-x_n)z + x_n] - f(x_n)\}/\{1 - f(x_n)\}.$$

Then for z on any closed bounded set in R_z we have uniformly $\lim_{n\to\infty} f_n(z) = 1-(1-z)^{\alpha/\pi}$. This result follows by Carathéodory's theory of mapping of variable regions, first for the case that R_w is symmetric in the real axis. A suitable transformation leads to results due to Lindelöf and Visser on the proportionality of angles at $z = 1$ and $w = 1$. These results, combined with a method similar to that of Carathéodory for mapping of angles bounded in part by straight lines, lead to the proof of the asymmetrical case. Some topological aspects and the case when R_w is not a Jordan region are discussed. The author's purpose is to indicate a method rather than to emphasize applications.

B. A. Amirà (Jerusalem).

Heinhold, Josef. Ein Schmiegungsverfahren der konformen Abbildung. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1948, 203–222 (1949).

The author describes an iteration method for the determination of the conformal map of a simply-connected domain D, containing the point at infinity, onto the outside of the unit circle. The difference between this method and that employed by Koebe in his proof of the Riemann mapping theorem [J. Reine Angew. Math. 145, 177–223 (1915)] is that Koebe's "root transformation" is replaced by another transformation of a similar character which, if judiciously applied, may accelerate the "smoothing out" of the boundary quite considerably. If the shape of the complement of D deviates strongly from that of a circle, a few steps often suffice to reduce it to nearly circular shape. Once this has been achieved, however, the convergence of the method

becomes much slower, and it is advisable to continue the approximation by one of the well-known methods for the conformal mapping of nearly circular domains.

Z. Nehari (St. Louis, Mo.).

Bazilevič, I. Ya. On distortion theorems for univalent functions. Uspehi Matem. Nauk (N.S.) **4**, no. 3(31), 128–130 (1949). (Russian)

Primarily an abstract of results to be found in the author's paper [Doklady Akad. Nauk SSSR (N.S.) **65**, 253–255 (1949); these Rev. **10**, 602]. *D. C. Spencer.*

Iliev, Lyubomir. On finite sums of univalent functions. Doklady Akad. Nauk SSSR (N.S.) **70**, 9–11 (1950). (Russian)

Let $f_k(z)=z+\sum_{n=1}^{\infty}a_{nk+1}z^{nk+1}$ be regular and univalent in $|z|<1$, and let $\sigma_n^{(1)}(z)$ be the sum of the first n terms of $f_1(z)$, $\sigma_n^{(2)}(z)$ the sum of the first n terms of $f_2(z)$. The author uses an inequality proved earlier [C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 1, 21–24 (1949); translation into Russian, Doklady Akad. Nauk SSSR (N.S.) **69**, 491–494 (1949); these Rev. **11**, 92], $|(f_k(z_1)-f_k(z_2))/(z_1-z_2)|\geqq(1-r^2)/(1+r^k)^{4/k}$ for $|z_1|, |z_2|<r$, to show that (I) for $n\geqq 15$, $\sigma_n^{(1)}(z)$ is univalent in $|z|<1-4n^{-1}\log n$; (II) for $n\geqq 4$, $\sigma_n^{(1)}(z)$ is univalent in $|z|<\frac{1}{4}$. He states without proof that (III) for $n\geqq 11$, $\sigma_n^{(2)}(z)$ is univalent in $|z|<(1-3n^{-1}\log n)^{\frac{1}{2}}$. Theorem (I) is an improvement of a result of Levin [Jber. Deutsch. Math. Verein **42**, 68–70 (1932)] and the author points out that by combining his proof of (II) with Szegö's proof for $n=2, 3$ one obtains a simpler proof of Szegö's theorem [Math. Ann. **100**, 188–211 (1928)] that for $n\geqq 2$, $\sigma_n^{(1)}(z)$ is univalent in $|z|<\frac{1}{4}$. *A. W. Goodman.*

Grad, Arthur. The region of values of the derivative of a schlicht function. Proc. Nat. Acad. Sci. U. S. A. **36**, 198–202 (1950).

The variational methods developed by A. C. Schaeffer and D. C. Spencer [Duke Math. J. **14**, 949–966 (1947); these Rev. **9**, 341] are used to determine the region $R(z_0)$ of values covered by $f'(z_0)$, $|z_0|<1$, when $f(z)$ ranges over the class $\mathfrak{S}$ of functions $f(z)=\sum_1^{\infty}a_nz^n$, $a_1=1$, which are regular and schlicht in $|z|<1$. There is no restriction on the problem if z_0 is taken to be a real number r, $0\leqq r<1$. The region $R(r)$ is found to be closed and bounded. It is symmetric by inversion in the circle $|z|=(1-r^2)^{-1}$ and expands continuously with increasing r. A function $f(z)$ of $\mathfrak{S}$ which omits a value w_0 for $|z|<1$ has its corresponding point $f'(r)$ interior to $R(r)$. It is shown that the boundary of $R(r)$ has a tangent everywhere. Although simply connected, $R(r)$ is not schlicht for $r>2^{-\frac{1}{2}}$. It is found desirable therefore to substitute for $R(r)$ the domain $L(r)$ of values at $z=r$ of the branch of $\log f'(z)$ which vanishes at $z=0$.

The differential equation of the functions $f(z)$ corresponding to the boundary of $L(r)$ is integrated to obtain the equation of the closed curve enclosing the region $L(r)$. For $0<r\leqq 2^{-\frac{1}{2}}$ this closed curve is analytic and any boundary function $w=f(z)$ maps $|z|<1$ onto the exterior of a single analytic slit in the w-plane extending to infinity. When $2^{-\frac{1}{2}}<r<1$, $L(r)$ is bounded by two arcs alternating with two straight-line segments. A boundary function corresponding to an interior point on these line segments maps $|z|<1$ onto the exterior of a slit which consists of a straight-line segment from a finite point to infinity together with two arcs, symmetrical with respect to the straight-line segment and meeting it at its finite end to form three equal angles. *M. S. Robertson* (New Brunswick, N. J.).

Goodman, A. W. On the Schwarz-Christoffel transformation and p-valent functions. Trans. Amer. Math. Soc. **68**, 204–223 (1950).

The transform applied to convex polygons on a Riemann surface is used to obtain inequalities for the coefficients, modulus, circle of univalence, circle of convexity, etc., of two classes of functions p-valent in a circle analogous to the convex and star-like schlicht functions. In particular, a few instances of a conjecture of the author [same Trans. **63**, 175–192 (1948); these Rev. **9**, 421] are established. In addition several special examples of the transformation, expressible in terms of elementary functions, are discussed in detail, and from properties of $\int_0^t t^{n\xi/2-1}dt/(1-t^n)^{\xi}$ for positive rational ξ a series of identities is deduced including the familiar $\pi^2=8\sum_0^{\infty}(2n+1)^{-2}$. *A. J. Macintyre.*

Havinson, S. Ya. On an extremal problem of the theory of analytic functions. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 158–159 (1949). (Russian)

Results are stated without proof. A function $f(z)$ analytic in the circle $|z|<1$ will be said to belong to class H_p if $\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\leqq M(f)$ for $0<r<1$, $p\geqq 1$. It is known then that $\lim_{r\to 1}f(re^{i\theta})=f(\xi)$, $\xi=e^{i\theta}$, for almost all ξ on the circumference $|\xi|=1$, where $f(\xi)$ is of class L_p. If the class H_p is taken to be a linear space with norm $\|f\|=\{\int_0^{2\pi}|f(\xi)|^p|d\xi|\}^{1/p}$, any linear functional $L(f)$ in the space H_p $(p>1)$ can be represented in the form $L(f)=\int_0^{2\pi}f(\xi)\omega(\xi)d\xi$, where $\omega(\xi)\varepsilon L_q$ $(1/p+1/q=1)$. The author states that the norm $\|L\|_{H_p}$ of the linear functional $L(f)$ is given by $\|L\|_{H_p}=\inf\{\int_0^{2\pi}|\omega(\xi)-\varphi(\xi)|^q|d\xi|\}^{1/q}$ for $\varphi\varepsilon H_q$. The greatest lower bound is attained for a unique function $\varphi^*(z)\varepsilon H_q$. A characteristic property of this function $\varphi^*(z)$ is the following. For almost all points on the circumference $|\xi|=1$ the function $f(\xi)=\xi^{-1}|\omega(\xi)-\varphi^*(\xi)|^q/(\omega(\xi)-\varphi^*(\xi))$ coincides with the boundary values of some function of class H_p. An analogous result is also stated for functions which are analytic and bounded in $|z|<1$. *W. Seidel.*

Shah, S. M. The maximum term of an entire series. V. J. Indian Math. Soc. (N.S.) **13**, 60–64 (1949).

[For part III cf. Quart. J. Math., Oxford Ser. **19**, 220–223 (1948); these Rev. **10**, 289. Part IV seems not to have appeared yet.] Soient $f(z)=\sum_0^{\infty}a_nz^n$ une fonction entière, $\mu(r)$ le maximum de $|a_n|r^n$ pour r fixé, $\nu(r)$ la valeur de n fournissant ce terme maximum, R_n défini par $\nu(r)=n$ si $R_n\leqq r<R_{n+1}$; l et L les lim inf et lim sup pour r infini de

$$n\left\{\frac{|a_n|^2}{|a_{n-1}|\,|a_{n+1}|}-1\right\}.$$

L'auteur montre que lim inf et lim sup de $\log|a_n/a_{n+1}|/\log n$ appartiennent au segment (l, L), que lim inf et lim sup de $[\nu(r)]^{-1}\log\mu(r)$ appartiennent au segment (a, b) formé par la lim inf a et par la lim sup b de $n\log(R_{n+1}/R_n)$ et que $a=\max(0, l)$, $b\leqq L$. *G. Valiron* (Paris).

Cypkin, Ya. Z. Stability of a class of systems of automatic regulation with distributed parameters. Avtomatika i Telemehanika **9**, 176–189 (1948). (Russian)

The stability, in the sense that certain quantities tend to zero as $t\to\infty$, hinges upon the location of the roots of $p(z)+q(z)e^{-\sigma z}=0$, where p and q are polynomials in z. Using familiar complex variable methods, various sufficient conditions for stability are given. Transcendental equations of the above type occur frequently in connection with differential-difference equations. *R. Bellman.*

★**Čebotarev, N. G., and Meĭman, N. N. The Routh-Hurwitz problem for polynomials and entire functions. Appendix by G. S. Barhin and A. N. Hovanskiĭ. Real quasipolynomials with $r=3$, $s=1$.** Trudy Mat. Inst. Steklov. **26**, 331 pp. (1949). (Russian)

The Routh-Hurwitz problem is the problem of finding conditions for the roots of a polynomial to lie in a half plane. This monograph contains two chapters by Čebotarev, one on the original Routh-Hurwitz problem (and generalizations to the case where a specified number of zeros lie in a half plane), and the other on the problem for quasipolynomials, which are expressions of the form $\sum_{n=1}^{m} a_n(z)e^{\lambda_n z}$ with real λ_n and polynomial $a_n(z)$. The rest of the book (except the appendix) is by Meĭman and deals with generalizations of the Routh-Hurwitz problem to entire functions, with some other results on the zeros of entire functions, and with preliminary material. Much of the material is more or less available in the periodical literature.

The first chapter is an account, mainly algebraic, and partly based on work by Yu. I. Neĭmark [summarized in Doklady Akad. Nauk SSSR (N.S.) **58**, 357–360 (1947); **59**, 853–856 (1948); these Rev. **9**, 348, 428], of the generalized Routh-Hurwitz problem for polynomials. A final section by Meĭman treats the Nyquist diagram, also making use of work of Neĭmark. Chapter 2 contains miscellaneous results from the general theory of functions. Chapter 3 is a concise introduction to the theory of entire functions in general and functions of exponential type in particular.

Meĭman's principal objective is to generalize the Hermite-Biehler theorem (that the roots of $g(z)+ih(z)$ are all on the same side of the real axis when $g(z)$ and $h(z)$ are polynomials with real coefficients, if and only if the zeros of g and h are real and interlacing), and the Hermite determinant criterion. It turns out that the most general class to which the Hermite-Biehler theorem can be extended is the class B of entire functions $F(z)=g(z)+ih(z)$, g and h real on the real axis, with $\limsup |F(z)/\bar{F}(z)| \leqq 1$ uniformly in x as $y\to+\infty$. The central theorem of chapter 4 [for detailed statements cf. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 46–49 (1943); these Rev. **6**, 59] gives necessary and sufficient conditions in terms of the Hadamard factorizations of g and h for $F(z)$ to belong to B and have all its zeros in a half plane (class HB). A number of further properties of the zeros of functions of classes B and HB and of their components g and h are given; some of these are described in some detail in a note by Meĭman [C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 11–14 (1946); these Rev. **8**, 323]. Two theorems are given on the representation of an entire function $F(z)$ which is nonnegative on the real axis as the square of the absolute value of a function of half the type; the first applies if the order of $F(z)$ is at least 1 and assumes the convergence of (*) $\sum |\sin\theta_n|/|a_n|$, where $a_n e^{i\theta_n}$ are the zeros of $F(z)$; the second was obtained independently by Ahiezer [Doklady Akad. Nauk SSSR (N.S.) **63**, 475–478 (1948); these Rev. **10**, 289]. Another result is that if $F(z)$ is of exponential type, the convergence of (*) makes the indicator diagram symmetric with respect to a line parallel to the real axis; the length of its intersection with the imaginary axis is specified in terms of the Hadamard factorization of $F(z)$.

Chapter 5 is devoted to determinant criteria for the zeros of entire functions, and particularly to Grommer's theorems [J. Reine Angew. Math. **144**, 114–166 (1914)]. Some of the results were given by Čebotarev [Math. Ann. **99**, 660–686 (1928)] with different proofs. The principal results of chapter 6, entitled "General formulation of the Routh-Hurwitz problem for entire functions," are summarized in a note by Meĭman [C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 179–181 (1943); these Rev. **6**, 59]. The chapter ends with a discussion of Nyquist diagrams for entire functions.

Chapter 7 on quasipolynomials begins by extending results of Pontryagin [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **6**, 115–134 (1942); these Rev. **4**, 214] from integral exponents λ_n to general λ_n. Attention is then restricted to functions $F(z)$ such that $F(iz)=g(z)+ih(z)$, where g and h are polynomials of the form $f(z, \cos z, \sin z)$, $f(z, u, v)$ having real coefficients and being of degree r in z and degree s in u and v. Sturm's method for polynomials is generalized. The cases $r=1$, $s=1$ and $r=2$, $s=1$ are studied in detail and the results expressed in explicit (but rather complicated) form [for the second case cf. Capyrin, Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 301–328 (1948); these Rev. **10**, 241]. A final section discusses technical applications. [On quasipolynomials in general, cf. also Neĭmark, Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 349–380 (1949); these Rev. **11**, 355]. The appendix [pp. 310–325] gives a similar discussion of the case $s=1$, $r=3$.

R. P. Boas, Jr. (Providence, R. I.).

Meĭman, N. On conditions under which the derivative of a majorant function is a majorant for the derivative of the function. Doklady Akad. Nauk SSSR (N.S.) **71**, 609–612 (1950). (Russian)

The author sketches proofs of theorems which extend Bernstein's theorem on the derivative of an entire function of exponential type to a much wider class of functions. Let $f(x)$ and $\omega(x)$ be two complex-valued functions of the real variable x, and introduce the conditions (A) $|f(x)| \leqq |\omega(x)|$, (B) for every real φ, $\arg\{e^{i\varphi}\omega(x)-f(x)\}$ is nondecreasing in a neighborhood of x. Under the assumptions that (A) and (B) hold in $[a, b]$, we have the following results. (1) Let $\theta(x)=\arg\omega(x)$; in any subinterval where $f(x)/\omega(x)$ is not constant, $\theta(x)$ is strictly increasing, and provided that $\theta(x_2)-\theta(x_1)\leqq\pi$ for $a\leqq x_1<x_2\leqq b$, we have

$$|\psi(x_2)-\psi(x_1)| \leqq 2\sin\tfrac{1}{2}\{\theta(x_2)-\theta(x_1)\}, \quad \psi(x)=f(x)/|\omega(x)|.$$

(2) Further, if f' and ω' exist at x, then $|f'(x)| \leqq |\omega'(x)|$. More generally, with $\cos\alpha=|f/\omega|'$, $\cos\beta=|\omega|'/|\omega'|$, $\gamma=\arg f'-\arg f$, we have

$$(*) \quad |f'|^2\sin^2\alpha+\{|f'|\cos\alpha\cos\gamma-|\omega'|\cos\beta\}^2 \leqq |\omega'|^2\sin^2\alpha.$$

(3) Further, if $f'(x)/f(x)$ is real, then according as it is positive or negative we have $|f'| \leqq |\omega'|\cos(\beta-\alpha)$ or $|f'| \leqq -|\omega'|\cos(\beta+\alpha)$. (4) Under the hypotheses of (1), $|\psi(x_2)-\psi(x_1)| \leqq \sin\{\theta(x_2)-\theta(x_1)\}$ if $f(x)$ is real in $[a, b]$. (5) As a converse to (2) and (3), it is stated that if f satisfies (A) at x and (*), or if f'/f is real and f satisfies (A) and the conclusion of (3) at x, then f satisfies (B).

The author now introduces the class HB of entire functions $F(z)=g(z)+ih(z)$, where g and h are real on the real axis, $F(z)$ and $\bar{F}(z)=g-ih$ have no common zeros and $|F(\bar{z})|<|F(z)|$ for $y<0$ ($z=x+iy$). Using results about class HB proved in the book reviewed above, he shows that, if f and ω are entire functions, $\omega\in HB$, $|f(x)| \leqq |\omega(x)|$ for real x, and one of ω/f, $\omega/\bar{f}$ does not have zero as an asymptotic value in the lower half plane, then in that half plane $|f/\omega|<1$, $|\bar{f}/\omega|<1$; for $|w|<1$ the function $\omega(z)-wf(z)$ belongs to HB; and as a consequence (B) is satisfied and the inequalities stated above are valid. In particular, this is true when f and ω are of exponential types τ, σ, $\tau\leqq\sigma$, and

$\omega \in HB$ [cf. the paper by Levin reviewed below]. When $\omega = e^{i\alpha}$ sup $|f(x)|$ we get Bernstein's theorem.

R. P. Boas, Jr. (Providence, R. I.).

Levin, B. Ya. On a special class of entire functions and on related extremal properties of entire functions of finite degree. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 45–84 (1950). (Russian)

The principal results of this paper were announced earlier [Doklady Akad. Nauk SSSR (N.S.) **65**, 605–608 (1949); these Rev. **11**, 23]. For generalizations cf. the preceding review. Some results concerning the Routh-Hurwitz problem for functions of exponential type were also announced earlier [C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 47–50, 99–100 (1943); these Rev. **6**, 59, 60], and are proved here; cf. also the second preceding review. *R. P. Boas, Jr.*

Levin, B. Ya. On entire functions of finite degree which are of regular growth. Doklady Akad. Nauk SSSR (N.S.) **71**, 601–604 (1950). (Russian)

The author starts with the following representation for a function $u(z)$ which is harmonic for $y>0$ and continuous for $y \geqq 0$: if $\limsup u(z)/|z| = \sigma < \infty$ and $\int_{-\infty}^{\infty}(t^2+1)^{-1}u(t)\,dt$ converges, then $\pi u(z) = xy\int_{-\infty}^{\infty} u(t)\{(t-x)^2+y^2\}^{-1}dt + ky$, where $k = \limsup y^{-1}u(iy)$. Corollaries: $\lim_{r\to\infty} r^{-1}u(re^{i\theta}) = k \sin\theta$ $(0<\theta<\pi)$; the same is true if $\int_0^{\infty} t^{-2}\{u(t)+u(-t)\}\,dt$ converges and $\limsup_{|x|\to\infty} |x|^{-1}u(x) \leqq 0$. The author calls $f(z)$, of exponential type (finite degree) in an angle, of completely regular growth if $\lim_{r\to\infty} r^{-1}\log|f(re^{i\theta})|$ exists in that angle except on a set of zero linear density. He proves that $f(z)$ is of completely regular growth in $y>0$ if it is of exponential type and either $\int_{-\infty}^{\infty}(1+t^2)^{-1}\log|f(t)|\,dt < \infty$ or if $\int_1^{\infty} t^{-2}\log|f(t)f(-t)|\,dt$ converges and $h(0)+h(\pi) \leqq 0$ (where h is the indicator function). He deduces two theorems about entire functions; he was evidently unaware that these had been obtained, by other methods, by Pfluger [Comment. Math. Helv. **16**, 1–18 (1944); these Rev. **5**, 258; see also Comment. Math. Helv. **18**, 177–203 (1946); these Rev. **7**, 426].

R. P. Boas, Jr. (Providence, R. I.).

Šteĭnberg, N. S. On Newtonian interpolation for entire functions. Doklady Akad. Nauk SSSR (N.S.) **71**, 21–22 (1950). (Russian)

The author states without proof several theorems sharpening the results of Ibragimov and Keldyš [Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 283–291 (1947); these Rev. **9**, 22] to the case $\theta = \frac{1}{2}$ [see the cited review for notation]. In particular, if $\lambda(r) \downarrow$, $r\lambda(r) \to 0$, $r[\frac{1}{2}-\lambda(r)] \uparrow$, there exist an entire $f(z)$ and a sequence of interpolation points such that $\lim n[r(\frac{1}{2}-\lambda(r))]/[r \log M(r)] = \infty$ and the Newton series for $f(z)$ diverges; if $\lambda(r)>0$, $r\lambda(r)\to\infty$, $r[\frac{1}{2}-\lambda(r)] \uparrow$, and $\limsup [r \log M(r)]/n[r(\frac{1}{2}-\lambda(r))] < \infty$, then the Newton series for $f(z)$ converges. Further results are stated under additional hypotheses on the interpolation points.

R. P. Boas, Jr. (Providence, R. I.).

Hornich, Hans. Beschränkte Integrale auf speziellen transcendenten Riemannschen Flächen. Monatsh. Math. **53**, 187–201 (1949).

The author considers two-sheeted Riemann surfaces F which have infinitely many real branch points $\alpha_0, \alpha_1, \alpha_2, \cdots$, $0<\alpha_n<\alpha_{n+1}$, $\alpha_n \to +\infty$ (and no other branch points). It is assumed that there are a real number $p>1$ and a sequence of positive numbers $q_n \to 0$ such that, for all n, $\alpha_{2n+1} \leqq p\alpha_{2n}$ and $\alpha_{2n+2} \geqq \alpha_{2n+1} \exp(-q_n\sqrt{\alpha_{2n+1}})$. Let β_n be a point of the interval $[\alpha_{2n}, \alpha_{2n+1}]$ and write

$$f_n(z) = (\alpha_{2n}\alpha_{2n+1}/\beta_n^2)(z-\beta_n)^2/(z-\alpha_{2n})(z-\alpha_{2n+1}).$$

The product $f(z) = \prod_0^{\infty} f_n(z)$ converges absolutely and uniformly in any compact subdomain which contains no branch point, and it defines a meromorphic function. The Riemann surface of $\sqrt{f(z)}$ is F. Let $\bar{F}$ be the surface F cut along the real axis from α_0 to $+\infty$. For distinct indices $i, k \geqq 0$, define

$$u_{ik} = \int_{\infty}^{z} \frac{\sqrt{f(z)}\,dz}{(z-\beta_i)(z-\beta_k)}.$$

It is shown that u_{ik} is bounded in $\bar{F}$, and that $\int |du_{ik}/dz| \cdot |dz|$ extended over a circle $|z| = R$ tends to zero as $R \to \infty$. It is proved that every bounded integral on $\bar{F}$ may be expressed as an infinite linear combination of suitably normalized integrals u_{ik}. *D. C. Spencer* (Princeton, N. J.).

Virtanen, K. I. Über Abelsche Integrale auf nullberandeten Riemannschen Flächen von unendlichem Geschlecht. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **56**, 44 pp. (1949).

Let F be a Riemann surface whose boundary has harmonic measure zero, a statement which is equivalent to the assumption that a single-valued harmonic function on F with a finite Dirichlet integral is a constant. Integrals w are considered which have only finitely many singularities and which satisfy the further condition that

$$D_{F-G}(w) = \iint_{F-G} |w'(z)|^2 dx\,dy < \infty$$

$(z = x+iy)$, where G is any compact subdomain of F containing all singularities of w in its interior. Let $A_1, B_1, \cdots, A_n, B_n, \cdots$ be a canonical (homology) basis for F, and let $P(w, A_n) = \int_{A_n} dw$, $P(w, B_n) = \int_{B_n} dw$. For each positive integer k an integral w_k of the first kind is constructed which satisfies the condition $P(w_k, A_n) = 0$ for $n = 1, 2, \cdots, k-1$, $P(w_k, A_n)$ real for $n \geqq k$, $P(w_k, B_n)$ real for $n \geqq k+1$. It is shown that the convergence of the series $\sum_k |\sum_{n=1}^{k}(a_n+ib_n)\bar{b}_n^{\,k}|^2$, $\bar{b}_n^{\,k} = \Im P(w_k, B_n)$, is necessary and sufficient in order that there should exist an integral w of the first kind with $P(w, A_n) = a_n+ib_n$, $n = 1, 2, \cdots$. This result is established by proving that the Dirichlet integral

$$D_F(w_1, w_2) = \iint_F \left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial y}\right) dx\,dy$$

is equal to

$$\sum_n \left(\int_{A_n} du_1 \int_{B_n} dv_2 - \int_{A_n} dv_2 \int_{B_n} du_1\right)$$

for any two integrals $w_1 = u_1+iv_1$, $w_2 = u_2+iv_2$ of the first kind for which either u_1 or v_2 has only finitely many non-vanishing A and B periods.

Integrals of the second and third kinds are also considered. It is proved that there exists an integral of the second kind having a single pole at an arbitrary point of F and having all its A periods zero. To each 1-chain L with boundary $a-b$ it is shown that there exists an integral $W_0(Z; L)$ of the third kind having singularities only at a, b with residues $+1, -1$ respectively such that its A periods vanish (provided that the integration path corresponding to each homology class A_n is chosen to avoid polar periods of $W_0(Z; L)$). A further section of the paper introduces a class of integrals of the first kind which can be constructed

by means of a convergent series of elementary normal integrals, and the concluding section is concerned with integrals of the first kind on a hyperelliptic surface.

D. C. Spencer (Princeton, N. J.).

Tsuji, Masatsugu. On the Green's function associated with a Fuchsian group. Jap. J. Math. **19**, 371–382 (1948).

Let G be a Fuchsian group of linear transformations which leaves $|z|=1$ invariant, and D_0 its fundamental domain containing $z=0$. It is assumed that D_0 is bounded by circular arcs orthogonal to $|z|=1$, and by a closed set e_0 on $|z|=1$ of positive measure. If a_n, $n=1, 2, \cdots$, denote the points equivalent to a point $a=a_0 \in D_0$ under the substitutions of G, the series

$$g(z, a)=\sum_{n=1}^{\infty} \log |(1-\bar{a}_n z)/(z-a_n)|$$

converges in $|z|<1$ and, for angular approach from within $|z|<1$, $g(z, a)$ tends to zero almost everywhere on $|z|=1$. The author calls $g(z, a)$ the Green's function of G because of the role it plays in the representation of an automorphic harmonic function $u(z)$, invariant under the substitutions of G, in terms of its boundary values at the points of e_0. The normal derivative of $g(z, a)$ on $|z|=1$ is shown to exist almost everywhere, and the formula

$$(1) \qquad u(a)=(2\pi)^{-1}\int_{e_0} u(e^{i\theta})\frac{\partial g(e^{i\theta}, a)}{\partial r}d\theta$$

is shown to hold. Like the Poisson formula, (1) can also be used to define a harmonic function invariant under the substitutions of G by means of given boundary values on e_0 for which the integral in (1) exists. For angular approach to $|z|=1$, the so defined harmonic function has almost everywhere on e_0 the prescribed boundary values.

Z. Nehari (St. Louis, Mo.).

See the note to p. 241, Tsuji.

***Reich, Theodor. Über das Verhalten einer regulären Quaternionenfunktion in der Nähe eines isolierten unwesentlich singulären Punktes oder einer isolierten unwesentlich singulären Kurve oder Fläche.** Thesis, University of Zürich, 1948. 44 pp.

Der Autor untersucht des Verhalten der im Fueter'schen Sinne regulären Quaternionenfunktionen in unwesentlich singulären Stellen. Er beweist in einem ersten Teil, dass auf jeder analytischen Kurve, auf der man in einen isolierten endlichen unwesentlich singulären Punkt einer solchen Funktion hineingeht, ein endlicher oder unendlicher Grenzwert existiert, und dass die Menge aller Grenzwerte, die zu allen möglichen in den betreffenden Punkt führenden analytischen Kurven gehören, niemals eine eindimensionale Mannigfaltigkeit bilden. Im zweiten Teil beweist er, dass auf jeder analytischen Kurve, auf der man in einen Punkt einer ganz im Endlichen liegenden unwesentlich singulären Punktmenge (deren Punkte, abgesehen von isolierten Punkten, auf isolierten Kurven und Flächen liegen und keine natürliche Grenze der Funktion sein sollen) einer regulären Quaternionenfunktion gelangt, immer ein endlicher oder unendlicher Grenzwert existiert, und dass auch hier die Menge aller Grenzwerte nie eine eindimensionale Mannigfaltigkeit sein können. Die Beweise stützen sich auf die Darstellung der regulären Quaternionenfunktionen, wie sie von Fueter und Nef entwickelt wurde. *H. G. Haefeli.*

Theory of Series

Lehner, Joseph. Note on the Laurent series. Amer. Math. Monthly **57**, 177–179 (1950).

The author considers a function $f(z)$ defined by a power series in z and having but one singular point, namely at $z=1$. His problem is to express the coefficients of the series expansion for $f(z)$ about $z=\infty$ in terms of the coefficients of the given series. He obtains rather complicated formulas involving doubly iterated infinite sums. *T. Fort.*

Toscano, Letterio. Sulla iterazione dell'operatore xD. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 337–350 (1949).

The author discusses the arrays of numbers defined by such equations as

$$(xD)^n=\sum_{i=0}^{n} k_{n+1,\,i+1}^{(\lambda)} x^{i+\lambda} D^i x^{-\lambda},$$

$$n!(xD)^n=\sum_{i=0}^{n} l_{n+1,\,i+1}^{(\lambda)} x^{n-i-\lambda+1} D^n x^{i+\lambda-1}.$$

Recursion formulas are readily obtained. In particular, for $\lambda=\frac{1}{2}$ the number $H_{n,i}=2^{n-1}l_{n,i}^{(\lambda)}$ has the symmetry property $H_{n,i}=H_{n,n-i+1}$. Connections with the Bernoulli and Euler numbers are indicated. *L. Carlitz* (Durham, N. C.).

Aczél, J. On some sequences defined by recurrence. Acta Univ. Szeged. Sect. Sci. Math. **13**, 136–139 (1949).

For a sequence defined by $a_n=\frac{1}{2}(a_{n-2}+a_{n-1})$, a_1, a_2 arbitrary, it is known that $\lim_{n\to\infty} a_n=(a_1+2a_2)/3$; consequently, for any quasi-arithmetic mean, if

$$a_n=f^{-1}\{\tfrac{1}{2}[f(a_{n-2})+f(a_{n-1})]\},$$

then $\lim_{n\to\infty} a_n=f^{-1}\{\frac{1}{3}[f(a_1)+2f(a_2)]\}$. The author considers the more general sequence defined by

$$(*) \qquad a_n=\left\{\sum_{j=1}^{k} p_j a_{n-k-1+j}\right\}\Big/\sum_{j=1}^{k} p_j,$$

where the p_j are positive and $a_1, \cdots, a_k$ are arbitrary, and shows that

$$\lim_{n\to\infty} a_n=\left\{\sum_{j=1}^{k} q_j a_j\right\}\Big/\sum_{j=1}^{k} q_j,$$

where $q_j=p_1+\cdots+p_j$. Indeed, for any sequence defined by $a_n=m(a_{n-k}, \cdots, a_{n-1})$, where m is consistent, strictly increasing, and continuous, it is shown that $\lim_{n\to\infty} a_n$ exists; this result holds even if m is not the same for every n, provided that all but at most a finite number of m's occur infinitely often. The well-known theorem of Eneström and Kakeya in the theory of algebraic equations is shown to be a consequence of the last of the above results.

E. F. Beckenbach (Los Angeles, Calif.).

Aczél, John. Some remarks on recurrent sequences. Nieuw Arch. Wiskunde (2) **23**, 144–149 (1950).

Generalizing (*) [see the preceding review], the author considers recurrent sequences determined by equations of the form $c_0 a_n+\cdots+c_k a_{n-k}+c=0$, where $c_0, \cdots, c_k, c, a_0, \cdots, a_k$ are given complex numbers. By means of linear transformations and the theory of linear difference equations, conditions on the c's are given for convergence and also for divergence of $\{a_n\}$. Alternative direct proofs are also given. Application is made to sequences determined by $F(a_n, \cdots, a_{n-k})=0$, where $F(z_0, \cdots, z_k)$ is an analytic complex function. *E. F. Beckenbach* (Los Angeles, Calif.).

Knopp, K. Beweis eines von I. Schur in der Theorie der C-Summierbarkeit aufgestellten Satzes. J. Reine Angew. Math. **187**, 70–74 (1949).

A sequence b_n is of type (C_j, C_k) [or (C_j', C_k)] if the series $\sum b_n a_n$ is evaluable C_k whenever $\sum a_n$ is evaluable C_j [or bounded C_j]. Let k be a real number and let j be an integer such that $0 \leqq k \leqq j$. Then b_n is of type (C_j, C_k) [or (C_j', C_k)] if and only if $b_n = O(n^{k-j})$ [or $b_n = o(n^{k-j})$] and $\sum n^j |\Delta^{j+1} b_n| < \infty$. As the author acknowledges in a footnote added in proof, these results are included in a contemporary paper of Bosanquet [Proc. London Math. Soc. (2) **50**, 482–496 (1949); these Rev. **10**, 368], where j is not restricted to integer values and different methods are used.

R. P. Agnew (Ithaca, N. Y.).

Orts, J. M.ª On the inversion of Euler's transformation. Revista Mat. Hisp.-Amer. (4) **9**, 154–158 (1949). (Spanish)

Let $\sum a_n z^n$ converge when $|z| < 1$, and let p be a positive integer. Let sequences b_n and c_n be defined by

$$\sum a_n \{(1-u^p)/(1+u^p)\}^n = \sum b_n (u^p-1)^n = \sum c_n (u-1)^n.$$

The problem discussed is that of expressing the elements of one of the sequences a_n, b_n, c_n in terms of the elements of another one. Explicit formulas are given for the case $p=2$. In the simplest case $p=1$, the formulas are known in the theory of Euler's transformation of series of order $r=1/(1+u)$. There are some misprints. *R. P. Agnew.*

Millar, J. G. Hyperbolic function series arising from a simple potential problem. Amer. Math. Monthly **57**, 100–104 (1950).

Identities are obtained by equating series solutions of a simple potential problem to the obvious unique solution, unity. *P. Franklin* (Cambridge, Mass.).

Fourier Series and Generalizations, Integral Transforms

Turán, P. On the distribution of real roots of almost-periodical polynomials. Publ. Math. Debrecen **1**, 38–41 (1949).

Let $H(x) = \sum_{\nu=0}^{n} a_\nu \cos \lambda_\nu x$, $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$, $a_\nu \geqq 0$. Denote by $N(H, a, d)$ the number of real zeros of $H(x)$ in the interval $(a-d, a)$. Then,

$$N(H, a, d) \leqq 6n \log \{2\gamma(|a|+d)/d\} + 6d\lambda_n.$$

A similar estimate can be obtained for $R(x) = \sum_{\nu=0}^{n} b_\nu \sin \lambda_\nu x$, $b_\nu \geqq 0$. The interest of these results is due to the fact that the estimates are independent of the coefficients a_ν and b_ν. The assumption of positivity of a_ν and b_ν is essential.

M. Kac (Ithaca, N. Y.).

Morse, Marston, and Transue, William. The Fréchet variation and a generalization for multiple Fourier series of the Jordan test. Rivista Mat. Univ. Parma **1**, 3–18 (1950).

This paper is one of a series of publications, by the same authors, devoted to the general program of improving and refining the theory of bilinear functionals and of multiple Fourier series. The basic idea is to replace the concept of bounded variation in the sense of Vitali, which played a fundamental role in the classical version of these theories, by the concept of bounded variation due to Fréchet [concerning this concept, see the authors, Canadian J. Math. **1**, 153–165 (1949); these Rev. **10**, 601]. The present paper is essentially an elaboration of a preliminary announcement [Proc. Nat. Acad. Sci. U. S. A. **35**, 395–399 (1949); these Rev. **11**, 19]. Beyond the result indicated by the title, the paper contains inequalities for the coefficients in multiple Fourier series. *T. Radó* (Columbus, Ohio).

***Carleman, T. Sur l'application de la théorie des fonctions analytiques dans la théorie des transformées de Fourier.** Analyse Harmonique, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, pp. 45–53. Centre National de la Recherche Scientifique, Paris, 1949. 600 francs.

Relations are found between certain classes of analytic functions which are analogous to those between a function of a real variable and its Fourier transform. These are used to prove the following theorem which, together with the underlying theory, is given in a more complete form in the author's book [L'intégrale de Fourier et questions que s'y rattachent, Uppsala, 1944; these Rev. **7**, 248]. If $k(x)$ is absolutely integrable in $(-\infty, \infty)$ and the only real zeros of its Fourier transform are numbers $\alpha_1, \alpha_2, \cdots$, with a finite exponent of convergence, the bounded solutions of $\int k(x-y)\phi(y)dy$ are the limit functions of uniformly bounded sequences of trigonometrical polynomials $\sum a_\nu e^{i\alpha_\nu x}$ converging almost everywhere. *H. R. Pitt* (Belfast).

Lebedev, N. N. On the representation of an arbitrary function by integrals involving cylinder functions of imaginary index and argument. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 465–476 (1949). (Russian)

The inversion formulae

$$(1) \qquad f(x) = 2x^{-1}\pi^{-2} \int_0^\infty \tau \sinh \pi\tau F(\tau) K_{i\tau}(x) d\tau,$$

$$(2) \qquad F(\tau) = \int_0^\infty f(x) K_{i\tau}(x) dx$$

given previously by the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 655–658 (1946); these Rev. **9**, 29] are here proved under wider conditions. If $f(x)$ is of bounded variation on every finite interval (a, b), and $f(x) \log x \in L(0, \frac{1}{2})$, $f(x)x^{\frac{1}{2}} \in L(\frac{1}{2}, \infty)$, then (1) holds with $[f(x+0)+f(x-0)]/2$ replacing $f(x)$. If in addition $f(x)$ is continuous in an interval the integral (1) converges uniformly in that interval. A further condition is stated without proof, reference being made to a proof of a similar theorem elsewhere. If $F(\mu/i)$ is an analytic function of $\mu = \sigma + i\tau$ holomorphic in $-\delta \leqq \sigma \leqq \delta$ and such that $F(-i\sigma+\tau)(\sigma+i\tau)e^{\frac{1}{2}\pi|\tau|} \in L(-\infty, \infty)$ and tends to 0 uniformly in σ as $\tau \to \infty$ for σ in $(-\delta, \delta)$, then if f is given by (1), (2) follows for all real τ. A number of instances of the formulae are given. *J. L. B. Cooper* (London).

Meixner, Josef. Über das asymptotische Verhalten von Funktionen, die durch Reihen nach Zylinderfunktionen dargestellt werden können. Math. Nachr. **3**, 9–13 (1949).

The author investigates the asymptotic behaviour of functions represented by Neumann series. More precisely, he proves that if the sequence $\{r!|\alpha_{\pm r}|\}^{1/r}$, $r = 1, 2, \cdots$ is bounded so that $f(u) = \sum_{\nu=-\infty}^{\infty} i^\nu \alpha_\nu H_{\nu+\tau}^{(1)}(u)$ converges for sufficiently large $|u|$, then the asymptotic expansion of $f(u)$ for $u \to \infty$ can be obtained by replacing each Bessel function by its asymptotic expansion and interchanging the order of summations. Similar results hold for other Bessel functions.

A. Erdélyi (Pasadena, Calif.).

Delerue, Paul. Sur l'utilisation des fonctions hyperbessé-liennes à la résolution d'une équation différentielle et au calcul symbolique à n variables. C. R. Acad. Sci. Paris 230, 912–914 (1950).

The author recalls the connection between the inverse operational images of $f(p)$ and of $p^{\nu-\mu}f(p^{-\tau})$, and the similar relation in two variables. The so-called hyperbessel functions occurring in these formulae are solutions of the differential equation $d^n y/dx^n + x^q y = 0$. *A. Erdélyi.*

***Labin, Édouard. Calcul Opérationnel.** Masson et Cie, Paris, 1949. iv+149 pp. 780 francs.

"Le présent fascicule constitue un avant-coureur d'un ouvrage plus vaste qui est en préparation . . .". This opening remark of the introduction justifies a closer scrutiny of the method of presentation than is normally due to yet one more addition to the already large and ever-growing literature on operational calculus. The whole work will be called "Abrégé théorique et pratique de mathématiques supérieures," and its purpose is to set forth all important mathematical techniques in a manner that will enable the physicist, engineer, etc. to apply the results to their problems. The author is very emphatic about his purpose being "renseignements d'action" rather than "enseignement de réflexion." The presentation is descriptive rather than deductive, proofs are rigorously banned, and the results are arranged in the order in which the practical user would expect to find them rather than in the logical order in which they arise in developing the theory. The material presented consists of formulas, theorems, and descriptive properties. While often a certain amount of circumlocution is useful, the author aims at a precise presentation of results together with their conditions of validity. The writing of such a work is a task of extraordinary difficulty, because it demands a combination of a thorough knowledge of both pure mathematics and its application with a rare gift of exposition and description; it is also a very attractive task, for a really good work constructed on such principles would be of the greatest importance to all who have to apply modern mathematical techniques. To say that the author has not fully succeeded in his task is saying the obvious; but a few detailed comments may be useful.

In a work in which results are given without proofs and without references, and the reader has no means of checking them, a scrupulously correct and precise statement of formulas and theorems is of paramount importance. The reviewer did not notice many misprints in the text and accompanying formulas, but in the tables the situation is different. In one of the principal tables giving 95 Laplace transform pairs, about one formula in every seven or eight is rendered incorrectly, and such an amount of misprints seriously inconveniences the user. Statements lacking in precision, and even incorrect statements, also occur in the book. In 1.–(4) it is stated that every Laplace transform (LT for short) has at least one singularity on its axis of convergence, while it is known that e.g. the LT of $\sin(e^t)$ is an entire function. Again, in 1.–(5) it is claimed that from the convergence of a Laplace integral no deductions can be made as to uniform convergence, while it is fairly elementary that every Laplace integral converges uniformly in any bounded closed domain in the interior of its half-plane of convergence, and better results are also known. In Table II the LT of $1-\operatorname{erf}[(\tau/4t)^{\frac{1}{2}}]$ for positive τ is given as $p^{-1}\exp\{-(\tau p)^{\frac{1}{2}}\}$ with the remark that $p=0$ is both a pole and a branch point of this function (!) and that any branch of $p^{\frac{1}{2}}$ may be taken (when only the principal branch leads at all to a LT). Several definitions of asymptotic expansions are given. The later ones are sound, but the first definition asserts that a series of increasing powers is an asymptotic expansion of a function $f(x)$ if for any positive integer n the ratio of the function and of the nth partial sum of the series tends to unity as x approaches zero. The reviewer is also disturbed by the use of standard mathematical terms in an unorthodox sense. For instance, $f(t)=O(g(t))$ is defined to mean that $f(t)$ and $g(t)$ are of the same order, "ou encore" that $f(t)/g(t)$ tend to a finite limit as t tends to 0 or ∞, as the case may be. Again, a function $f(t)$ is first said to be of exponential order if for some real a it is $o(e^{at})$ as $t\to\infty$, then the "more precise" definition is put forward that there be a real number a such that $f(t)e^{-at}$ is integrable at ∞ for all $x>a$, and fails to be integrable for all $x<a$. Worst of all, the symbol $\subset$ is used to indicate the correspondence between $f(t)$ and its LT $F(p)=\int_0^\infty e^{-pt}f(t)dt$, whereas this symbol was introduced by MacLachlan, and is exclusively used to-day, to indicate the correspondence between $f(t)$ and $pF(p)$.

Bibliography is another item of great importance. Here one feels that perhaps too many papers have been quoted, not all of them very important, and certainly not enough books. For instance, Carslaw and Jaeger are represented by one paper each, but their excellent book on the application of the Laplace transformation is missing; one paper by Churchill is quoted, but not his book, the best book on operational calculus from the point of view of the engineer and physicist; Doetsch's theoretical treatise is in the list, but not his all-important Tabellen zur Laplace-Transformation und Anleitung zum Gebrauch [Springer-Verlag, Berlin-Göttingen, 1947; these Rev. 9, 237]; there is an expository paper by Widder in the bibliography, but one looks in vain for his book; and Titchmarsh is missing altogether. Apart from Volterra's lectures on composition, the Italian school is entirely disregarded. The desire not to quote too many books in foreign languages is understandable, and the reviewer is not complaining about the number of references to literature in English or German, only about their selection.

This criticism of details should not be regarded as a wholesale condemnation of the work. On the contrary, the fact that a comparatively few details could be singled out for criticism shows that on the whole the author has succeeded better than one had any right to expect. He must be congratulated on a very courageous and meritorious enterprise. In a second edition, or translation, it should be possible, perhaps with the assistance of a mathematician, to remove the blemishes and to realise fully the potential usefulness of the work. Paper and printing are very good, and although the format is slightly unusual and does not seem to be motivated by the needs of this particular volume, it may have been chosen with regard to the needs of other volumes of this series.

An indication of the contents of this volume, with a few specific comments, follows. I. Definitions and generalities. Existence, convergence, inversion, connection with the Fourier transformation. II. Basic properties. Simple operations with LT's, jump function, product theorem, function of a function, iterated LT. III. A first table of LT's. Thirteen LT's which are rational functions of p, multiplied in some cases by an exponential function. General rules for the interpretation of rational functions of p. IV. Evaluation of the complex inversion formula by the calculus of residues in the case of one-valued LT's. V. Many-valued

LT's. A table of 6 LT pairs of this class. VI. Relations between the behaviour of $f(t)$ at 0 and ∞, and the behaviour of $F(p)$ at ∞ and 0. Simple Tauberian theorems are included, but not Abelian theorems for the inverse Laplace transformation. VII. Series expansions. Convergent and asymptotic power series, and Dirichlet series (in p). VIII. Singular functions. The delta function and its derivatives. It is one of the best features of this book that it segregates singular functions, and makes it clear that the general theorems of the first seven chapters are not necessarily valid for such functions. IX. List of LT's; 95 LT pairs arranged according to the $F(p)$, and 116 LT pairs arranged according to the $f(t)$. Among the latter are 10 jump functions. X. Applications of the Laplace transformation to ordinary differential equations with constant coefficients, simple integro-differential equations and systems of such equations. Ordinary differential equations with variable coefficients are not included. XI. Applications to partial differential equations. The formal process is described. The need for justification is mentioned, but no hints in this direction are given, nor is there any reference to help the reader. XII. Integral equations of the composition type. XIII. Recurrence relations and difference equations. Bibliography (3½ pp.), index, table of contents. *A. Erdélyi.*

Goodspeed, F. M. The Mellin transform of functions defined by infinite series. Trans. Roy. Soc. Canada. Sect. III. (3) **43**, 15–20 (1949).

Let $\Phi(x)$ be the analytic continuation of

$$\varphi(0)-x\varphi(1)+x^2\varphi(2)-\cdots.$$

G. H. Hardy [Ramanujan, Cambridge University Press; Macmillan, New York, 1940, chapter 11; these Rev. 3, 71] gave conditions for $\varphi(s)$, under which Ramanujan's "interpolation" formula

$$\int_0^\infty x^{s-1}\Phi(x)dx=\pi\varphi(-s)/\sin\pi s$$

is valid. The author replaces $\sin\pi s$ by $C(s)=\prod(1-s^2/\lambda_n^2)$, $(0<A\leqq\lambda_n/n\leqq B)$, and can then determine a function $\psi(s)$ from its values at the points λ_n. It is assumed that $\psi(s)$ is of exponential type in $R(s)\geqq\delta$ $(\delta<\lambda_1)$, and of uniformly smaller growth than $C(s)$ on the lines $R(s)=\sigma\geqq\delta$. Then

$$\int_0^\infty x^{s-1}\Psi(x)dx=\psi(-s)/C(s),\quad -\lambda_1<\Re(s)<\min(\lambda_1,-\delta),$$

where $\Psi(x)$ is the analytic continuation of the sum of the series $-\sum\psi(\lambda_n)\{C'(\lambda_n)\}^{-1}x^{\lambda_n}$, which converges (for small x) after bracketing. The proof is obtained by applying Mellin's inversion formula to the integral

$$(2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty}x^{-s}\{\psi(-s)/C(s)\}ds=\Psi(x),$$

$x>0$; $-\lambda_1<\sigma<\min(\lambda_1,-\delta)$. *J. Korevaar.*

Polynomials, Polynomial Approximations

Peters, Johannes. Wann gilt das Stabilitätskriterium nach Nyquist? Arch. Elektr. Übertragung **4**, 17–22 (1950).

For networks with active and passive parts, stability as determined from negative real parts of roots is translated into a condition on certain paths in the complex plane which slightly generalizes the condition of Nyquist. The Nyquist condition is always sufficient, and is equivalent to the generalized condition in several practically important special cases. *P. Franklin* (Cambridge, Mass.).

Demontvignier, Marcel, et Lefèvre, Paul. Une nouvelle méthode harmonique d'étude de la stabilité des systèmes linéaires. Rev. Gén. Électricité **58**, 263–279 (1949).

An exposition is given of methods of testing stability of linear systems, from Routh to Nyquist. A simple method of constructing phase diagrams and of deducing stability from them applicable to the majority of practical cases is explained. *P. Franklin* (Cambridge, Mass.).

Haag, Jules. Sur la stabilité des points invariants d'une transformation. Bull. Sci. Math. (2) **73**, 123–134 (1949).

Considering a rather restricted type of transformation the author proves certain results which he erroneously says lead to the theorem that a necessary and sufficient condition for stability is that the characteristic roots of the matrix of the linear terms of the transformation be less than 1 in magnitude. [The transformation $TX=X-X^3$, where X is a real variable, is stable near $X=0$, since $T^nX\to0$ as $n\to\infty$ for all small X, and yet the linear part is simply $LX=X$. Thus the condition is not necessary. That it is sufficient is a classical result.] Other aspects are considered.

N. Levinson (Cambridge, Mass.).

Grad, Harold. Note on N-dimensional Hermite polynomials. Comm. Pure Appl. Math. **2**, 325–330 (1949).

Orthogonal polynomials in N variables can be formed as products of N such polynomials each in a single variable. In the author's opinion "such a procedure lacks symmetry, and there is sometimes an advantage to be gained by expressing the polynomials in tensor invariant notation." The weight function is $\omega=(2\pi)^{-N/2}\exp(-\frac{1}{2}\sum x_ix_i)$, and all tensorial indices run from 1 to N. Divested of the special notation introduced in this paper, the polynomials in question are

$$(-)^n\omega^{-1}\frac{\partial^n}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_n}}\omega,\quad i_1,\cdots,i_n=1,\cdots,N,$$

and thus clearly products of ordinary Hermite polynomials. There is no reference to the extensive literature on multidimensional Hermite polynomials [cf. Appell and Kampé de Fériet, Fonctions hypergéométriques et hypersphériques, Polynomes d'Hermite, Gauthier-Villars, Paris, 1926, and later papers]. *A. Erdélyi* (Pasadena, Calif.).

Olovyanišnikov, V. M. An estimate for the remainder in approximation, best on a given system of points, of continuous periodic functions by polynomials. Doklady Akad. Nauk SSSR (N.S.) **70**, 761–764 (1950). (Russian)

Let H_ω be the class of all continuous functions $f(x)$ of period 2π having a prescribed modulus of continuity $\omega(\delta)$, so that $|f(x')-f(x)|\leqq\omega(|x'-x|)$ for each $f\in H_\omega$. Let $E_{T_n}(H_\omega,x)=\sup|f(x)-T_n(x)|$ for $f\in H_\omega$, where $T_n(x)$ is the polynomial of order $n-1$ giving best approximation of f at the system of $2n$ equidistant points $x_k=k\pi/n$ $(k=1,2,\cdots,2n)$. (Since T_n has $2n-1<2n$ coefficients, T_n in general does not interpolate f at the points x_k.) The author shows that

$$E_{T_n}(H_\omega,x)=\pi^{-1}\omega(\pi/n)|\sin nx|\log n+O(\omega(n^{-1})),$$
$$E_{T_n}(H_\omega,k\pi/n)=\tfrac{1}{2}\omega(\pi/n).$$

Corresponding formulas are obtained for functions f having an $(r-1)$th derivative satisfying a Lipschitz condition of

order 1. There is close analogy between the above formulas and the formulas of Nikolsky [C. R. (Doklady) Acad. Sci. URSS (N.S.) 31, 215–218 (1941); these Rev. 2, 360] giving the order of approximation of f by the trigonometric polynomials of order n interpolating f at $2n+1$ equidistant points. *A. Zygmund* (Chicago, Ill.).

Olovyanišnikov, V. M. An estimate for the remainder in the polynomial approximation, best on a given sequence of points, to nonperiodic functions satisfying a Lipschitz condition. Doklady Akad. Nauk SSSR (N.S.) 71, 613–616 (1950). (Russian)

Let KH denote the class of functions $f(x)$, $-1\leqq x\leqq 1$, satisfying the condition $|f(x_1)-f(x_2)|\leqq K|x_1-x_2|$. For any $f\varepsilon KH$, let $P_n(f,x)$ be the polynomial of degree $n-1$ giving best approximation to f on the system of points $x_k=\cos(k\pi/n)$ $(k=0, 1, \cdots, n)$. Let $E_{P_n}(KH,x)=\sup|f(x)-P_n(f,x)|$, $f\varepsilon KH$. It is shown that

$$E_{P_n}(KH,x)=2K\pi^{-1}\sin|(\pi/2n)(1-x^2)^{\frac{1}{2}}\sin n\arccos x|\log n+O(n^{-1}),$$

$E_{P_n}(KH,\cos(k\pi/n))=K_n^{-1}$. Corresponding results are obtained for even functions of the class KH, and for polynomials of degree $2n-2$ giving best approximation to f at the points $\sin(k\pi/2n)$. *A. Zygmund* (Chicago, Ill.).

Harmonic Functions, Potential Theory

Walsh, J. L., Sewell, W. E., and Elliott, H. M. On the degree of polynomial approximation to harmonic and analytic functions. Trans. Amer. Math. Soc. 67, 381–420 (1949).

Let $u(z)$ $(z=x+iy)$ be harmonic in the interior of the Jordan curve C and continuous in the closed region $\bar{C}$, let $w=\phi(z)$ be a conformal map of the exterior of C on $|w|>1$ so that $\phi(\infty)=\infty$, and let C_R be the image of $|w|=R$ (>1) in the z-plane. If $u(z)$ is harmonic in C_R, then there are harmonic polynomials $p_n(z)$ (degree $n=1, 2, \cdots$) such that (1) $\limsup_{n\to\infty}[\max|p_n(z)-u(z)|$ on $C]^{1/n}\leqq 1/R$. Conversely, if $u(z)$ is suitably defined exterior to C and if, for some sequence $\{p_n(z)\}$, (1) holds, then $u(z)$ is harmonic in C_R. This result, due to Walsh, is extended and sharpened in the present paper. A theory is developed analogous to that of the approximation to analytic functions $f(z)$ by polynomials $\pi_n(z)$ [see Sewell, Degree of Approximation by Polynomials in the Complex Domain, Princeton University Press, 1942; these Rev. 4, 78], incidentally sharpening results of the latter theory. For continuity properties of $\Re[f(z)]=u(z)$ only and not of $f(z)$, or degrees of approximation to $u(z)$ only occur in the assumptions, while results are deduced for $f(z)$. Naturally this requires an extension of the Riesz theory of conjugate functions from $|z|=1$ to Jordan curves. This and other auxiliary results are of interest in themselves.

The authors deal with three problems [cf. Sewell loc. cit.]: the relation between the degree of convergence in $\bar{C}$ and continuity properties of $u(z)$ on C (problem α) or on C_R (problem β), and the behaviour of approximating polynomials on C_R (problem γ). They start with preliminary results concerning one- and two-dimensional derivatives on C, and Lipschitz conditions. Then they deal with approximation on the unit circle, making use of the partial sums of the Fourier series of $u(re^{i\theta})$. Employing polynomial inequalities and conformal transformations, they prove their main results; on (α), for instance, imposing some conditions on C: if $\partial^k u/\partial s^k\varepsilon\mathrm{Lip}\,\alpha$ on C $(k=0, 1, \cdots; s=\text{arc length})$, then there are polynomials $\pi_n(z)$ such that $|f(z)-\pi_n(z)|\leqq Mn^{-k-\alpha}$ (z in $\bar{C}$). Conversely if there are harmonic polynomials $p_n(z)$ such that $|u(z)-p_n(z)|\leqq Mn^{-k-\alpha}$ (z in $\bar{C}$; $k+\alpha\geqq 1$), then $f^{(k)}(z)\varepsilon\mathrm{Lip}\,\alpha$ on $\bar{C}$ for $0<\alpha<1$,

$$|f^{(k)}(z_2)-f^{(k)}(z_1)|\leqq M'|z_2-z_1||\log|z_2-z_1||$$

for $\alpha=1$ (z_1, z_2 on C); while, for $0<k+\alpha<1$ and under some other conditions on C, $f(z)\varepsilon\mathrm{Lip}\,\alpha$ on C. Also they deal with (β) for the segment $-1\leqq z\leqq 1$, taking C_R as the ellipse $|z+1|+|z-1|=R+1/R$, $z=\phi^{-1}(w)=\frac{1}{2}(w+1/w)$, $|w|=R>1$, and with (β) for negative k. Finally they show by examples that some of their theorems, concerning the critical case $\alpha=1$ and a related one, are best possible results. *H. Kober* (Birmingham).

Walsh, J. L., and Elliott, H. Margaret. Polynomial approximation to harmonic and analytic functions: generalized continuity conditions. Trans. Amer. Math. Soc. 68, 183–203 (1950).

The class of continuous and periodic functions $f(\theta)$ such that $|f(\theta+h)+f(\theta-h)-2f(\theta)|\leqq M|h^\alpha|$ $(M=M(f,\alpha))$ may be denoted by $\mathrm{Zg}\,\alpha$, say. A. Zygmund [Duke Math. J. 12, 47–76 (1945); these Rev. 7, 60] has shown that this class and, for periodic $f(\theta)$, $\mathrm{Lip}\,\alpha$ are identical when $0<\alpha<1$; that they are not for $\alpha=1$, while $\mathrm{Lip}\,1\subset\mathrm{Zg}\,1$; that the case $\alpha=1$ ceases to be exceptional in results, for instance, on conjugate functions and on degree of approximation by trigonometric polynomials, if $\mathrm{Lip}\,\alpha$ is replaced by $\mathrm{Zg}\,\alpha$. Now in similar problems the case $\alpha=1$ plays a particular part in a recent paper by W. E. Sewell and the authors [see the preceding review for the following results and notations]. Their present paper is based on Zygmund's results. They start with the definition (1.1): Z_A^k is the set of functions $f(z)$ which are analytic in C, continuous on $\bar{C}$ and such that $F(s)\varepsilon\mathrm{Zg}\,1$ where s is on C, $f^{(k)}(z)=F(s)$, $s=$arc-length. The definition (2.1) of the class Z^k of harmonic functions is similar. A number of preliminary results are deduced, for instance: $f(z)\varepsilon Z_A^k$ on $|z|=1$ if, and only if, $f(z)$ is analytic for $|z|<1$ and, for $z=re^{i\theta}$ and $r<1$, $f^{(k+1)}(z)<L/(1-r)$ (theorem 2.1); when C is analytic then $f(z)\varepsilon Z_A^k$ on C if, and only if, $\Re[f(z)]\varepsilon Z^k$ on C. On their main problems (α) and (β) the authors prove: if $\Re[f(z)]\varepsilon Z^k$ on C, which may consist of several mutually exterior analytic Jordan curves, then, for some sequence $\{\pi_n(z)\}$ and z on $\bar{C}$, $|f(z)-\pi_n(z)|<L/n^{k+1}$; there are two converse theorems. When C is subject to some condition and $u(z)$ is defined on $\bar{C}$, then the existence of $\{p_n(z)\}$, with $|u(z)-p_n(z)|\leqq M/n^{k+2}R^n$ (z on $\bar{C}$, $R>1$), implies that $f(z)\varepsilon Z_A^k$ on C_R, where $\Re[f(z)]=u(z)$. This result is extended to negative values of k; the definition (1.1), however, is modified for $k<0$, in connection with theorem 2.1. Finally results on the segment $-1\leqq z\leqq 1$ are deduced. *H. Kober* (Birmingham).

Schiffer, M., and Szegö, G. Virtual mass and polarization. Trans. Amer. Math. Soc. 67, 130–205 (1949).

Dans l'espace ordinaire la perturbation d'un courant liquide (à densité constante) de direction $\mathfrak{h}(h_1, h_2, h_3)$ (vecteur-unité) par immersion d'un solide se traduit par un courant additif dont le potentiel des vitesses $\Phi=h_1\Phi_1+h_2\Phi_2+h_3\Phi_3$ est essentiellement d'énergie $W\int|\mathrm{grad}\,\Phi|^2d\tau$ (étendue à l'extérieur). De même la perturbation d'un champ électrique de direction $\mathfrak{h}$ par un solide conducteur se traduit par un champ additif de potentiel Ψ avec énergie P analogue à W;

Φ et Ψ se comportent à l'infini comme un dipôle, mais à la surface du solide $d\Phi/dn=-\mathbf{h}\cdot\mathbf{n}$, $\Psi=\mathbf{h}\cdot\mathbf{r}+$constante ($\mathbf{n}$ normale, $\mathbf{r}$ rayon vecteur). On appelle W et P, qui sont des formes quadratiques par rapport aux h_i, masse virtuelle et polarisation dans la direction $\mathbf{h}$. Les auteurs se proposent essentiellement d'étudier, en relation avec les caractères géométriques du solide, les invariants de ces formes et surtout la masse virtuelle moyenne W_m et la polarisation moyenne P_m définies respectivement comme moyennes des coefficients des termes carrés. Cette étude ressemble à celle récente de Pólya et Szegö sur la capacité [Amer. J. Math. 67, 1–32 (1945); ces Rev. 6, 227 et articles ultérieurs]; si elle ne conduit pas encore à des résultats aussi précis, elle donne beaucoup d'inégalités comme $PW\geqq V^2$, $P_mW_m\geqq V^2$ (V volume). Des outils essentiels sont un produit scalaire de fonctions harmoniques, comme l'intégrale du produit scalaire des gradients, les fonctions de Green, Neumann et leurs variations (Hadamard), la fonction noyau étudiée par Bergman et Schiffer pour des équations plus générales de type elliptique [Duke Math. J. 14, 609–638 (1947); ces Rev. 9, 187]. De nombreux exemples sont approfondis, dont celui des solides presque sphériques. *M. Brelot.*

Nevanlinna, Rolf. Über Mittelwerte von Potentialfunktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 57, 12 pp. (1949).

Let the open Riemann surface F be exhausted by a sequence of compact regions $F_0\subset F_1\subset\cdots\subset F_n\subset\cdots$, $F_n\to F$, where $F-F_0$ is connected and the boundary of F_n is composed of finitely many analytic arcs γ_n. The analytic arcs γ_n separate γ_0 from the ideal boundary Γ of F, and $\gamma_n\to\Gamma$ as $n\to\infty$. The author supposes that Γ has harmonic measure zero; that is to say, if ω_n is the harmonic measure of γ_n with respect to the region F_n-F_0, then $\lim\omega_n=0$. If u is a regular single-valued harmonic function in $(F-F_0)+\gamma_0$ with finite Dirichlet integral, he proves that $\int_{\gamma_n}u(\partial u/\partial n)ds\to 0$ as $n\to\infty$. As a corollary it follows that every single-valued harmonic function on F with a finite Dirichlet integral is a constant. Moreover, a single-valued harmonic function in $F-F_0$ with finite Dirichlet integral is uniquely determined by its boundary values on γ_0. Thus harmonic functions with finite Dirichlet integrals behave as though the boundary Γ of F were not present. *D. C. Spencer.*

Parreau, Michel. La théorie du potentiel sur les surfaces de Riemann à frontière positive. C. R. Acad. Sci. Paris 230, 914–916 (1950).

The author outlines an extension of the results of classical potential theory for a Riemann surface with positive boundary (Riesz representation theorem, capacity, balayage).

M. Heins (Providence, R. I.).

Differential Equations

Hartman, Philip, and Wintner, Aurel. On the classical existence theorem of linear differential equations. Amer. J. Math. 71, 859–864 (1949).

Une fonction $f(s)$ est uniformément presque-périodique pour $\sigma\geqq 0$ si elle est uniformément presque-périodique pour $\sigma>0$ (étant holomorphe dans ce demi-plan ouvert) et si la fonction limite $f(it)$ est continue et uniformément presque-périodique pour $-\infty<t<\infty$ (la presque-périodicité de $f(\sigma+it)$ par rapport à t étant uniforme par rapport à σ lorsque $\sigma\to+0$). Soit (1) $dw_i/ds=\sum_{k=1}^{n}f_{ik}(s)w_k$ ($i=1, 2, \cdots, n$). Si les n^2 fonctions f_{ik} sont uniformément presque-périodiques pour $\sigma\geqq 0$ avec $f_{ik}(s)\sim\sum_m a_m^{(i,k)}l^{-\lambda_m s}$, borne inférieure $\lambda_m>0$, toute composante w_i d'une solution $(w_1, w_2, \cdots, w_n)$ de (1) est uniformément presque-périodique pour $\sigma\geqq 0$. Les n constantes d'intégration C_i peuvent être choisies comme termes constants des développements de Fourier des w_i, et tout ensemble de n valeurs moyennes C_i définit une solution $(w_1, \cdots, w_n)$ d'une manière unique. Les exposants non-nuls des w_i sont des combinaisons linéaires à coefficients entiers positifs d'un nombre fini des λ_n. *S. Mandelbrojt* (Paris).

Coddington, Earl A., and Wintner, Aurel. On the classical existence theorem of analytic differential equations. Amer. J. Math. 71, 886–892 (1949).

Soit $\alpha(x)$ une fonction à variation bornée sur $[1, \infty)$. On désignera par $[\alpha]$ la variation totale de α. Soient $f_1, \cdots, f_n$, n fonctions holomorphes lorsque $\Re s>0$, $|\omega_i|<r$ ($i=1, \cdots, n$), données sous la forme

$$f_i=f_i(s;\omega_1, \cdots, \omega_n)=\sum_{k_1, \cdots, k_n=0}^{\infty}a_i^{(k_1, \cdots, k_n)}(s)\omega_1^{k_1}\cdots\omega_n^{k_n},$$

avec $a_i^{(k_1, \cdots, k_n)}(s)=\int_1^\infty e^{-sx}d\alpha_1^{(k_1, \cdots, k_n)}(x)$,

$$\sum[\alpha_i^{(k_1, \cdots, k_n)}]r^{k_1+\cdots+k_n}<\infty$$

($i=1, \cdots, n$). Il existe une constante $l\geqq 0$ telle que le système $dw_i/ds=f_i$ ($i=1, \cdots, n$) admette une solution unique pouvant se mettre sous la forme $w_i(s)=\int_1^\infty e^{-sx}d\beta_i(x)$ ($i=1, \cdots, n$), ces intégrales étant uniformément convergentes dans le semi-plan $\Re s\geqq l$. *S. Mandelbrojt* (Paris).

Ghurye, S. G. The case of complex roots in the method of Frobenius. J. Univ. Bombay (N.S.) 18, part 3, sect. A, 1–3 (1949).

The series solutions for complex roots of the indicial equation are expressed in terms of real functions.

P. Franklin (Cambridge, Mass.).

Hukuhara, Masuo. Sur la généralisation des théorèmes de M. J. Malmquist. J. Fac. Sci. Univ. Tokyo. Sect. I. 6, 77–84 (1949).

J. Malmquist has treated the equation $x^\alpha y^\beta y'=f(x, y)$, where α, β are positive integers, in the neighborhood of (0, 0) [Acta Math. 74, 175–196 (1941); these Rev. 3, 240]. The present paper gives existence theorems generalizing his results to the case where α, β are real constants satisfying simply $1\leqq\alpha$, $-1<\beta$. *J. M. Thomas* (Durham, N. C.).

Bylov, B. F. On an estimate for the characteristic numbers of the solution of almost diagonal systems of linear differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 114–116 (1950). (Russian)

The paper deals with a regular system

$$x_i'=\sum p_{ij}(t)x_j, \qquad i, j=1, \cdots, n,$$

with coefficients continuous and bounded for $t\geqq 0$ and such that $p_{ii}\geqq p_{i+1, i+1}+2(n-1)\{p_{jk}\,|\,j\neq k\}$. For such a system there are given rather complicated bounds for the characteristic numbers in the sense of Liapounoff.

S. Lefschetz (Princeton, N. J.).

Gusarov, L. A. On the approach to zero of the solutions of a linear differential equation of the second order. Doklady Akad. Nauk SSSR (N.S.) 71, 9–12 (1950). (Russian)

It is shown that all solutions of $y''+p(x)y=0$ approach zero as $x\to\infty$, provided that $p'(x)$ is a continuous function

of bounded variation for $x \geqq x_0$, and $q(x) \leqq p(x) \leqq q(x)^s$, where s is an integer greater than one, and $q(x)$ is monotone and $\to\infty$ as $x\to\infty$. If $y''+r(x)y=0$, $\int^\infty |r(x)-p(x)|\,dx<\infty$, then all solutions approach zero as $x\to\infty$. These criteria cover cases not included in previous results of Armellini and Wiman. *R. Bellman* (Stanford University, Calif.).

Putnam, C. R. On isolated eigenfunctions associated with bounded potentials. Amer. J. Math. 72, 135–147 (1950).

Real solutions of $x''+(\lambda+f(t))x=0$, $0\leqq t<\infty$, where $f(t)$ is continuous and $|f(t)|<C$, are considered subject to the boundary condition $x(0)\cos\alpha+x'(0)\sin\alpha=0$. The spectrum is denoted by $S(\alpha)$ and the derived spectrum by S'. Let λ satisfy $\lambda+\liminf f(t)>0$ or else $\lambda+\limsup f(t)<0$ (where lim is taken as $t\to\infty$). If

$$\limsup \int_0^t (x^2(s)+x'^2(s))ds/(x^2(t)+x'^2(t)) = \infty,$$

then λ is in $S(\alpha)$. If $x(t)$ is $L^2(0,\infty)$ and if

$$\limsup \int_t^\infty (x^2(s)+x'^2(s))ds/(x^2(t)+x'^2(t)) = \infty,$$

then λ is in S'. If $x(t)$ is $L^2(0,\infty)$ and λ is not in S', then $x(t)$ and $x'(t)$ are exponentially small as $t\to\infty$.

N. Levinson (Cambridge, Mass.).

Rubinowicz, A. The limits of the applicability of Sommerfeld's polynomial method in quantum theory. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 40 (1947), 57–63 (1948).

The author discusses the conditions under which the Sturm-Liouville problem

$$\frac{d}{dx}\left(p\frac{df}{dx}\right)-qf+\lambda\rho f=0$$

may be solved by the form $f=EP$, where P is a polynomial and q is independent of λ. The results are then applied to the one-dimensional Schroedinger equation.

H. Feshbach (Cambridge, Mass.).

Terracini, Alessandro. Su una proprietà differenziale conforme di certi sistemi triplamente infiniti di curve. Univ. e Politecnico Torino. Rend. Sem. Mat. 8, 227–240 (1949).

This paper is concerned with geometrical characterizations of families of curves defined by differential equations of the general form

$$(1)\qquad y'''=F(x,y,y')+\{3y'/(1+y'^2)\}y''^2,$$

and of the special form obtained by setting

$$F=(1+y'^2)[-v(x,y)-2u(x,y)y'+v(x,y)y'^2],$$

where u and v are conjugate harmonic functions. Let Φ denote a three-parameter family of plane curves, and let Φ_{AB} denote the one-parameter subfamily composed of those curves which pass through both of two points A, B. Let m_1 and m_2 denote the slopes of the tangents to the typical curve of Φ_{AB} at A and B, respectively. If, for every pair of neighboring points A, B, there exists a constant k such that one of the relations

$$(2)\qquad m_2(1-km_1)=k+m_1,$$
$$(3)\qquad m_2(1+km_1)=k-m_1$$

is satisfied, the author says that the family Φ has the property of equality. It is shown that the families of curves having this property are the families which are obtainable from the family of circles in the plane by conformal transformations.

The equation (2) is of no further interest. From the equation (3) there is obtained the differential equation

$$(4)\qquad (1+m_1^2)dm_2/dm_1+(1+m_2^2)=0.$$

Let the abscissae of A and B be denoted by x_0 and x_0+h, respectively, and develop the left-hand member of (4) in powers of h. Usually the expansion begins with a term in h^2. If this term is absent, so also is the term in h^4, and the expansion usually begins with a term in h^5. In this case the author says that the family Φ has the property of equality to at least the first approximation, and he shows that Φ is defined by an equation of the form (1). If the term in h^5 is also absent, the family Φ has the property of equality exactly, and the defining differential equation is of the special form described above. These characterizations of the families of curves are compared with and related to various other characterizations which have been given by the present author and by others. *L. A. MacColl.*

Noto, Silvia. Proprietà geometriche delle equazioni differenziali del tipo

$$y^{IV}=A(x,y,y',y'')y'''^2+B(x,y,y',y'')y'''+C(x,y,y',y'').$$

Univ. e Politecnico Torino. Rend. Sem. Mat. 8, 209–221 (1949).

It has been shown by Kasner that the families of plane curves defined by differential equations of the form

$$(1)\quad y^{iv}=A(x,y,y',y'')y'''^2+B(x,y,y',y'')y'''+C(x,y,y',y'')$$

are characterized by the following property. The centers of the ∞^1 conics which osculate the curves passing through a point P in a direction y', and with a common curvature, lie on a conic which passes through P in the direction y'. Such families of curves are called systems of type Q. In this paper the author states this property in projective terms, and relates it to some alternative characteristic properties. A mapping of the plane upon an arbitrary surface carries a system of plane curves of type Q into a family of curves which is defined by an equation of the form (1), where x and y are now coordinates on the surface. Such nonplanar families of curves are also called systems of type Q. The latter part of the paper is devoted to a study of projections which carry a system of type Q on a surface into certain somewhat specialized systems of type Q in a plane.

L. A. MacColl (New York, N. Y.).

Teghem, J. Sur des séries entières en le signe de dérivation D. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 1042–1053 (1949).

Let $C_n=a_0b_n+\cdots+a_nb_0$. The author studies the validity of the equation

$$(\sum a_iD^{i+p})(\sum b_kD^{k+q})=\sum C_nD^{n+p+q},$$

where each series is summed by the modified Euler method introduced by the author in his thesis [Brussels, 1946; these Rev. 10, 112]. Equality is given the interpretation customary in discussing differential operators. *R. C. Buck.*

✱**James, Hubert M., Nichols, Nathaniel B., and Phillips, Ralph S., editors. Theory of Servomechanisms.** Massachusetts Institute of Technology, Radiation Laboratory Series, vol. 25. McGraw-Hill Book Company, Inc., New York and London, 1947. xiv+375 pp. $5.00.

The past ten years have witnessed a fundamental advance in the theory of servomechanisms and of automatic control in general. Whereas the older theory amounted to little

more than a study of the motions of particular systems from the standpoint of classical mechanics, the newer theory is based upon the broad concept of control as being essentially a matter of the transmission of signals, and it utilizes the highly special methods of communication engineering. This new theory, which is limited in its applicability to linear systems, has been expounded in several recent books. Of these, the one reviewed here is easily the most comprehensive and definitive. Different parts of the book are written by different authors, of which there are ten in all. As a result, all phases of the subject are treated with a high degree of authority. The editing has been done so competently that, despite the multiplicity of authors, the unity and coherence of the work are generally quite satisfactory.

The opening chapter gives an excellent introductory survey of the entire subject of automatic control. This is followed by a chapter on the basic mathematical theory of linear systems. The discussion deals with the theories of steady-state and transient phenomena, and with the stability of systems having feedback. Chapter 3 discusses the elements, such as motors, synchros, etc., out of which servomechanisms are constructed. Unlike the other chapters, this one requires considerable prior technical knowledge on the part of the reader. However, the chapter contains a large amount of specific information which will be valuable to the properly equipped engineer. Chapter 4, which runs to nearly a hundred pages, deals with the design of servomechanisms to meet specified requirements. The discussion is accompanied by a large amount of illustrative material, in the way of design charts, examples, etc., and it appears to be entirely adequate for most purposes. The fifth chapter is devoted to a discussion of servomechanisms which operate with pulsed (as opposed to continuous) data. In the final three chapters the point of view changes. The input signal is supposed to be, in some sense, a random function, and the system is supposed to be subjected to extraneous noise, which is also described statistically. The problem under consideration is that of designing the system so that the error, in a suitably defined statistical sense, shall be a minimum. The delicate mathematical theory which is involved here is presented as rigorously as the general purpose of the book permits, and in a form which should appeal strongly to engineers and physicists.

Except for a few minor details of exposition in chapter 2, the reviewer has no fault to find with anything in the book. However, it is unfortunate that the authors have limited themselves so strictly to linear systems. It is true that at the present time it is not possible to give anything in the way of a general theory of nonlinear servomechanisms. Nevertheless, servomechanisms which are more or less nonlinear, either by accident or design, are important and of frequent occurrence, and the theoretical discussion of such systems is not entirely out of the question. If the authors and editors had seen fit to include a chapter in which some of these nonlinearities were discussed as far as our present knowledge permits, what is already an excellent treatise would have been made still stronger. *L. A. MacColl.*

*Hänny, Jost. **Über die Berechnung der Reguliervorgänge in linearen Systemen.** Thesis, Eidgenössische Technische Hochschule in Zürich, 1947. ii+254 pp.

This book is a clearly and competently written introduction to the modern theory of linear servomechanisms and regulators. A general descriptive survey of the subject is given in the introduction. Chapter I is devoted to a discussion of the stability of feedback systems, the discussion centering around the Nyquist stability criterion. This arrangement and treatment of the material testifies to the author's sound understanding of the fundamentals of the subject. The responses of servomechanisms to given input signals are treated in chapter II; and in chapter III the author sets up the fundamental equations of motion for a number of particular systems, mostly regulators to control the speeds of turbines. A short fourth chapter gives a résumé of the Laplace transformation theory which is used in the calculation of transient responses. Fundamentally, the book is quite similar to several others which have appeared in the U. S. A. recently. It differs from most of the other books in that it places less emphasis upon general theory, and more upon the properties of particular systems, such as the regulators mentioned above. The author is very much interested in procedures of analytical and numerical computation, and he discusses these at great, perhaps too great, length. *L. A. MacColl* (New York, N. Y.).

Ulanov, G. M. On the maximum deviation of the regulated quantity in a transient process. Avtomatika i Telemehanika **9**, 168–175 (1948). (Russian)

The mathematical problem is that of determining the maximum of $x(t)$ where x satisfies the linear operational equation $f(p)x=y$ $(p=d/dt)$, and we are given that $|y|\leqq a$. Using the standard Laplace transform method of solution, the author discusses a number of special cases, giving bounds for x in terms of a and $f(p)$. *R. Bellman.*

Rubbert, F. K. Der Einfluss der Dämpfung bei nichtlinearen Schwingungen. I. Gedämpfte Pendelschwingungen. Ing.-Arch. **17**, 336–342 (1949).

Treatment of several types of damping by familiar formal processes but with no consideration of convergence.
N. Levinson (Cambridge, Mass.).

*Vettin. **On the integration of non-linear differential equations. Zur Integration nichtlinearer Differentialgleichungen.** Lockheed Aircraft Corporation, Burbank, Calif., 1947. i+15 pp. (2 plates)+i+20 pp.

[Translation and reproduction of Zentrale für Wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB), Forschungsber. 1795/1.] Expressions of the form (1) $y(x)=\sum_{k=1}^{m}c_k e^{r_k x}$ are obtained as approximations to the solution of the general nonlinear equation (2) $f(x, y, y', \cdots, y^{(p)})=0$, having initial values $y^{(\nu)}(0)=y_\nu$, $\nu=0, 1, \cdots, p-1$. The parameters c_k, r_k are determined by equating the νth derivatives at $x=0$ of the left and right members of (1). Here ν runs from 0 to $2m-1$; for values of ν beyond $p-1$ the derivatives $y^{(\nu)}(0)$ are obtained by repeated differentiation of (2). The r_k turn out to be the roots of a characteristic equation

$$\begin{vmatrix} y_0 & y_1 & \cdots & y_{m-1} & 1 \\ y_1 & y_2 & \cdots & y_m & r \\ \cdots & & & \cdots & \\ y_m & y_{m+1} & \cdots & y_{2m-1} & r^m \end{vmatrix}=0.$$

The convergence of the process is not discussed.
J. Wendel (New Haven, Conn.).

Lighthill, M. J. A technique for rendering approximate solutions to physical problems uniformly valid. Philos. Mag. (7) **40**, 1179–1201 (1949).

A heuristic discussion of a procedure for solving nonlinear ordinary or partial differential equations containing

a small parameter in a critical region of the independent variable, that is, a region where the convergence of familiar processes breaks down as the region is approached. The author's device, illustrated by several applications, is to expand the independent variable as well as the dependent variable in a power series of the small parameter with the coefficients functions of a new auxiliary variable. The formal procedure for obtaining the coefficients is indicated. In one application the author observes that the method is simply the Poincaré procedure for small parameters.

N. Levinson (Cambridge, Mass.).

Krasovskiĭ, A. A. On a vibrational method for linearizing some nonlinear systems. Avtomatika i Telemehanika 9, 20–29 (1948). (Russian)

In the appendix the author uses Bogoliuboff's theorem [On Some Statistical Methods in Mathematical Physics, Akad. Nauk Ukrainskoĭ SSR, 1945; these Rev. 8, 37] to prove that the solutions of

$$(1)\quad x_k=\psi_k(t)+F_k(x_1+\varphi_1(\omega t),\ x_2+\varphi_2(\omega t),\ \cdots,\ x_n+\varphi_n(\omega t)),$$

$k=1,\cdots,n$, are approximated arbitrarily well for large ω by the solutions of

$$(2)\quad y_k=(2\pi)^{-1}\omega\int_0^{2\pi/\omega}[\psi_k(t)+F_k(y_1+\varphi_1(\xi),\cdots,y_n+\varphi_n(\xi)]d\xi,$$

$k=1,\cdots,n$. The φ_k are continuous and periodic with period 2π; the F_k are continuous and satisfy a Lipschitz condition in a suitable region. The main body of the work applies this result to an equation arising in a mechanical problem involving dry friction: $a\ddot{x}+b\dot{x}+cx+\varphi(\dot{x})=p(t)+A\sin\omega t$, where $\varphi(\xi)=(M/q)\xi$ for $|\xi|\leqq q$, $\varphi(\xi)=M$ $(-M)$ for $\xi>q$ $(\xi<-q)$.

J. Wendel (New Haven, Conn.).

Karreman, George. Some types of relaxation oscillations as models of all-or-none phenomena. Bull. Math. Biophys. 11, 311–318 (1949).

Heuristic consideration of two equations of the form $\ddot{x}+\epsilon f(x)\dot{x}+x=0$:

$$(1)\quad f(x)=(x-p)(x-q),\qquad 0<p<q;$$

$$(2)\quad f(x)=(x^2-1)(x^2-9).$$

J. Wendel (New Haven, Conn.).

Kac, A. M. On the approximate solution of nonlinear differential equations of the second order. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 111–113 (1950). (Russian)

The equation dealt with is

$$(1)\quad \ddot{x}+h(x,\dot{x})\dot{x}+k^2(x,\dot{x})x=0$$

of importance in many applications to mechanics and radio. It includes among others van der Pol's equation. In general only approximate solutions are possible. The main interest is in nearly periodic solutions:

$$(2)\quad x=\alpha\sin\left(\int_0^t\omega dt+\epsilon\right).$$

To make $\dot{x}$ have the same form as if α, ω, ϵ were constant one imposes

$$\dot{\alpha}\sin\left(\int_0^t\omega dt+\epsilon\right)+\alpha\epsilon\cos\left(\int_0^t\omega dt+\epsilon\right)=0.$$

Assuming first ω constant and following van der Pol, α and ϵ slowly varying functions of t (i.e., $\dot{\alpha}$, $\dot{\epsilon}$ small) one obtains the approximate system integrable by quadratures:

$$(3)\quad \dot{\alpha}=-\frac{\alpha}{2\pi}\int_0^{2\pi}h(\alpha\sin u,\ \alpha\omega\cos u)\cos^2 u\,du,$$

$$(4)\quad \dot{\epsilon}=\frac{1}{2\pi\omega}\int_0^{2\pi}K^2(\alpha\sin u,\ \alpha\omega\cos u)\sin^2 u\,du-\tfrac{1}{2}\omega.$$

If ω constant and ϵ small are inadequate assumptions one may assume ω variable but $\dot{\omega}$ small and the average of $\dot{\epsilon}$ over a period zero. One obtains first (4) and also (3) but with $\dot{\alpha}$ replaced by $\dot{\alpha}+\frac{1}{2}\alpha\dot{\omega}/\omega$. From this one derives the system

$$\omega^2=\pi^{-1}\int_0^{2\pi}K^2(\alpha\sin u,\ \alpha\omega\cos u)\sin^2 u\,du,$$

$$\dot{\alpha}\left(1+\frac{\alpha}{2\omega}\frac{d\omega}{d\alpha}\right)=-\frac{\alpha}{2\pi}\int_0^{2\pi}h(\alpha\sin u,\ \alpha\omega\cos u)\cos^2 u\,du,$$

which may also be integrated by quadrature. As an example, a certain regulator with nonlinear servomotor gives rise to the equation

$$2T\ddot{\sigma}+3|\sigma|^{\frac{1}{3}}\dot{\sigma}+2|\sigma|^{\frac{1}{3}}\sigma=0$$

with the approximate solution

$$\sigma=\frac{24\cdot 2T^2}{(t+C)^2}\sin\{3(2(t+C))^{\frac{1}{3}}+\epsilon\},$$

where C is an arbitrary constant. The system has then an oscillation with frequency and amplitude both tending to 0 whatever C may be.

S. Lefschetz (Princeton, N. J.).

***Raševskiĭ, P. K. Geometričeskaya Teoriya Uravneniĭ s Častnymi Proizvodnymi. [Geometrical Theory of Partial Differential Equations.]** OGIZ, Moscow-Leningrad, 1947. 354 pp.

The title of this book does not reflect its contents, for only a small part of it is devoted to partial differential equations. It is a book of considerable content and covers many fields. The first two chapters are devoted to exterior forms and exterior differentiation. Conditions on the coefficients are carefully and clearly stated and some of the integral theorems are derived. The names of Grassmann, Plücker, Cartan, who have done so much for this subject, are not mentioned. The next two chapters deal with Pfaffian systems, their characteristic elements and with the integrals of completely integral systems. Four sets of necessary and sufficient conditions for complete integrability are derived (theorem of Frobenius). Chapters V, VI and VII deal with the class of a system of Pfaffian equations and of Pfaffian forms. The various relations between the rank and class of a system and the rank of the characteristic system are discussed; the canonical forms are derived. These results are applied to a single first order partial differential equation and the complete integral of Lagrange is obtained. Chapters VIII and IX give a very thorough treatment of a single Pfaffian equation of odd class; here the Poisson brackets are introduced and a generalized form of Jacobi's identity is derived. Chapter X is devoted to some applications such as the variational problem for a Finsler space, equations of dynamics, etc. The final chapter is devoted to a system of partial differential equations of general order. By regarding the partial derivatives as new dependent variables the problem is reduced to finding the integral varieties of a Pfaffian system lying in the subspace defined by the differential equations. The treatment is complete and accurate and limitations on its validity precisely stated.

On the whole it is a good text book, but not a treatise. There is no bibliography. *M. S. Knebelman.*

Petrovskiĭ, I. G. On some problems of the theory of partial differential equations. Amer. Math. Soc. Translation no. 12, 43 pp. (1949).

Translated from Uspehi Matem. Nauk (N.S.) 1, no. 3–4 (13–14), 44–70 (1946); these Rev. 10, 301.

Persidskiĭ, K. P. On an infinite system of partial differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 23–44 (1950). (Russian)

The author considers the nonlinear system of partial differential equations

$$\partial z_r/\partial t+\sum_s \frac{\partial z_r}{\partial x_s}\left[\sum_{k=1}^{\infty} p_{sk}x_k+L_s(t,x)\right] = \sum_{p=1}^{\infty} q_{rp}z_p+\sum_{k=1}^{\infty}\omega_{rk}x_k+N_r(t,x,z),$$

$r=1, 2, \cdots$. The problem is equivalent to that of solving an infinite system of ordinary differential equations. This is accomplished by converting this system into a system of integral equations, and using the method of successive approximations. *R. Bellman.*

Manfredi, Bianca. Decomposizione in prodotto di operazioni elementari delle espressioni alle derivate parziali, del primo ordine e totalmente lineari. Boll. Un. Mat. Ital. (3) 4, 381–390 (1949).

Based on n particular solutions of the differential equation

$$\partial z/\partial x_0+\sum_1^n a_i(x_0, x_1, \cdots, x_n)\partial z/\partial x_i=0,$$

the author presents an operational method for finding the general solution of the equation

$$\partial z/\partial x_0+\sum_1^n a_i\partial z/\partial x_i+b(x_0, x_1, \cdots, x_n)z=f(x_0, x_1, \cdots, x_n).$$

F. G. Dressel (Durham, N. C.).

Pogorelov, A. V. A priori estimates for the derivatives of a regular solution of a partial differential equation of elliptic type. Uspehi Matem. Nauk (N.S.) 4, no. 4(32), 179–182 (1949). (Russian)

A second order partial differential equation of elliptic type is defined on a two-sphere, if an elliptic equation $F_G(r, s, t, p, q, z, u, v)=0$ ($r=z_{uu}$, $s=z_{uv}$, $t=z_{vv}$, $p=z_u$, $q=z_v$) is given in every neighborhood G in which there are defined analytic coordinates u, v, and if in the (nonempty) intersection $G_1\cap G_2$ the coordinate change $(u_1, v_1)\to(u_2, v_2)$ transforms $F_{G_1}=0$ into $F_{G_2}=0$. Let $f(X)$, X a point on the sphere, satisfy such an equation. In the neighborhood of X let u, v be the geographical coordinates determined by choosing a great circle γ passing through X as the equator. Let $F_{\gamma,X}=0$ be the equation written in these coordinates. Set

$$f_{\gamma(i,j)}(X)=\left.\frac{\partial^{i+j}f(u,v)}{\partial u^i\partial v^j}\right|_{u=v=0},$$

$$F_{\gamma,X(m_1,\cdots,m_k)}=\left.\frac{\partial^{m_1+\cdots+m_k}F_{\gamma,X}(r,\cdots,v)}{\partial r^{m_1}\cdots\partial v^{m_k}}\right|_{r=f_{\gamma(2,0)}(X),\cdots}.$$

The author describes a modification of the method due to S. Bernstein [Comm. Soc. Math. Kharkow (2) 11, 1–164 (1909)] which permits one to find upper bounds for the moduli of $f_{\gamma(i,j)}(X)$, $i+j\geqq 0$, depending on upper bounds of the moduli of the functions $F_{\gamma,X(m_1,\cdots,m_k)}$. *L. Bers.*

Višik, M. I. The method of orthogonal and direct decomposition in the theory of elliptic differential equations. Mat. Sbornik N.S. 25(67), 189–234 (1949). (Russian)

Generalizing the work of H. Weyl [Duke Math. J. 7, 411–444 (1940); these Rev. 2, 202] to higher order elliptic equations, the author considers among others the differential equation

$$(1)\quad Lu(x)=\sum_{i_1,\cdots,i_m=1}^{n}\frac{\partial^{2m}u(x)}{\partial x_{i_1}^2\cdots\partial x_{i_m}^2} -\sum\theta_{k_1\cdots k_{m-1}}\frac{\partial^{2m-2}u(x)}{\partial x_{k_1}^2\cdots\partial x_{k_{m-1}}^2}+\cdots+(-1)^{m-1}\sum\theta_i\frac{\partial^2u(x)}{\partial x_i^2} +(-1)^m\theta_0u(x)=0$$

in some open region G in real n-space. The numbers $\theta_{k_1\cdots k_l}$ ($l=0, \cdots, m-1$) are symmetric in their indices and take the values 0 or 1. With (1) is associated a Hilbert space H whose elements are sets of real functions

$$(h)=(h_{i_1\cdots i_m}(x), \theta_{k_1\cdots k_{m-1}}h_{k_1\cdots k_{m-1}}(x), \cdots, \theta_0h(x))$$

which are symmetric in their indices and square integrable in G. The scalar product is

$$((h), (g))=\int_G(\sum h_{i_1\cdots i_m}(x)g_{i_1\cdots i_m}(x) +\sum\theta_{k_1\cdots k_{m-1}}h_{k_1\cdots k_{m-1}}(x)g_{k_1\cdots k_{m-1}}(x)+\cdots+\theta_0h(x)g(x))dx.$$

The space H has a subspace ψ^0 whose elements are of class C_m in G, vanish outside a compact subset of G and satisfy the differential equation

$$(2)\quad \sum\frac{\partial^m\psi_{i_1\cdots i_m}(x)}{\partial x_{i_1}\cdots\partial x_{i_m}}-\sum\theta_{k_1\cdots k_{m-1}}\frac{\partial^{m-1}\psi_{k_1\cdots k_{m-1}}(x)}{\partial x_{k_1}\cdots\partial x_{k_{m-1}}}+\cdots +(-1)^m\theta_0\psi(x)=0$$

in G. Let ψ' be the closure of ψ^0. Let F be the subspace of H consisting of elements (f) such that $f_{k_1\cdots k_l}(x)$, $l=0, \cdots, m$, are the weak derivatives $\partial^l f(x)/\partial x_{k_1}\cdots\partial x_{k_l}$ of a square integrable function f in G. Let Z^0 be the subspace of F consisting of elements (f) for which the corresponding function f is of class C_{2m} in G and vanishes outside a compact subset of G and let Z be the closure of Z^0. Let U be the orthogonal complement of Z in F. The author proves that, $\oplus$ denoting direct sum, $H=\psi'\oplus U\oplus Z$. The elements of $\psi=\psi'\oplus U$ are in a certain sense solutions of (2) and an element (u) of U is a weak solution of (1) so that $\int u(x)Lf(x)dx=0$ whenever f is of class C_{2m} in G and vanishes outside a compact subset of G. Using the elementary solution of (1) constructed by E. E. Levi it is possible to show that every weak solution is almost everywhere equal to an ordinary solution so that U is in fact a space of ordinary solutions. In a certain sense the derivatives of order less than m of an element of Z tend to zero at the boundary of G and hence the decomposition (3) $F=U\oplus Z$ gives the solution of the analogue of Dirichlet's problem for (1). In a similar way the decomposition $\psi=U\oplus\psi'$ gives the solution of the analogue of Neumann's problem.

The following variant of (3), the "direct decomposition," is given. If g is of class C_{2m} in G and satisfies a certain condition (automatically satisfied by solutions of (1), and if $n>2m$ given by $\int_G|Lg(x)|^{2n/(n+2m)}dx<\infty$), then $g=u+f$ where u is analytic in G, $Lu=0$, and $f\in Z$. Finally, the author establishes the existence of a solution of (1) with given periods. *L. Gårding.*

Kriszten, Adolf. Elliptische Systeme von partiellen Differentialgleichungen mit konstanten Koeffizienten. Comment. Math. Helv. **23**, 243–271 (1949).

The considerations of this paper are along the lines of the Zürich papers on hypercomplex functions, which were initiated by R. Fueter. The author considers systems of partial differential equations with constant coefficients

$$(1)\qquad \sum_{h=1}^{n}\sum_{i=0}^{m-1} a_{hk}^{(i)}\partial u_h/\partial x_i+\sum_{h=1}^{n} b_{hk}u_h=0,\quad k=1,\cdots,n,$$

having the property that each u_i satisfies the same second order equation with constant coefficients

$$(2)\qquad \left(\sum_{h,k=0}^{m-1} g_{hk}\partial^2/\partial x_h\partial x_k+2\sum_{h=0}^{m-1} d_h\partial/\partial x_h+c\right)u_i=0.$$

The system (1) is called elliptic or hyperbolic according to whether equation (2) is elliptic or hyperbolic. The main objective of the paper is to generalize to elliptic systems (1) the power series expansion theorems of ordinary complex variable theory. A normal form for the elliptic systems (1) is given, for which the corresponding equation (2) is simply

$$(3)\qquad \left(\sum_{h=0}^{m-1}\partial^2/\partial x_h^2+\omega^2\right)u_i=0,$$

where ω is a real or purely imaginary constant. For hyperbolic systems (1), H. Malmheden [Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] **8**, 1–116 (1947); these Rev. **9**, 440] has given a corresponding canonical form and introduced the notion of a multiplier of a system (1). If (1) is written in matrix form

$$(4)\qquad \left(\sum_{i=0}^{m-1} A_i\partial/\partial x_i+B\right)u=0,$$

then an operator M is called a multiplier of (4) provided that the symbolic matrix $M\cdot(\sum_{i=0}^{m-1}A_i\partial/\partial x_i+B)$ is of diagonal form. The author shows that $M=\sum_{i=0}^{m-1}A_i^*\partial/\partial x_i+B^*$ is a (first order) multiplier of a system (4) if and only if the m matrices A_i generate a Clifford algebra. Once this Clifford algebra is introduced, the system (1) is assumed to be of normal form, and suitable definitions of analytic functions are introduced [patterned after A. Kriszten, same Comment. **20**, 333–365 (1947); these Rev. **9**, 355]. Analogues of Cauchy's theorem, Cauchy's formula, and Taylor's expansion of ordinary complex variable theory are then proved. A chief source of complications is that in equation (3), ω need not be zero. *J. B. Diaz.*

Gårding, Lars. Le problème de Dirichlet pour les équations aux dérivées partielles elliptiques homogènes à coefficients constants. C. R. Acad. Sci. Paris **230**, 1030–1032 (1950).

The author outlines a method of solution for the Dirichlet problem for the general elliptic linear homogeneous differential equation with constant coefficients of order $2m$ in n independent variables. Let $g(x_1,\cdots,x_n)$ denote a function which is m-times differentiable in a closed set S and for which

$$\int_S\sum(Dg)^2dx_1\cdots dx_n<\infty,$$

where the sum is extended over all derivatives Dg of g of order m. The Dirichlet problem solved consists in the determination of a solution u of the differential equation, for which in a certain generalized sense the derivatives of $g-u$ of order less than m vanish on the boundary of S.

F. John (New York, N. Y.).

Brousse, Pierre. Sur quelques propriétés d'une équation du type elliptique, rencontrée en élasticité. C. R. Acad. Sci. Paris **230**, 713–714 (1950).

Let $A=(-d,1)$, $B=(d,1)$, $d>0$, be two points of the (x,y)-plane, ASB a continuous arc joining A and B and lying in the "rectangle" $-d\leqq x\leqq d$, $0<y\leqq 1$, the point $S=(0,1-p)$ being the point of minimum ordinate on the arc. Let Σ be the curve consisting of the two half lines $(x\geqq d, y=1)$, $(x\leqq -d, y=1)$, plus the arc ASB, and D the (unbounded) connected open set whose boundary consists of Σ and the x-axis. The author considers the boundary value problem P of Dirichlet type: to determine a regular function $u(x,y)$, bounded at infinity, satisfying $\Delta u=3y^{-1}\partial u/\partial y$ in D, where $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$, and assuming prescribed boundary values on the boundary of D. It is stated that by considering the function $u(x,y)\pm\epsilon\cosh x\varphi(y)$; $\epsilon>0$, a constant, and choosing the function φ in a suitable way, it may be shown that zero is the only solution of the homogeneous problem P with zero boundary values, hence P has at most one solution. A related boundary value problem is also considered. *J. B. Diaz.*

Miles, John W. A note on Riemann's method applied to the diffusion equation. Quart. Appl. Math. **8**, 95–101 (1950).

The equations considered are of the form

$$(1)\qquad L(u)=\frac{\partial}{\partial x}\left[x^{m-1}\frac{\partial}{\partial x}u(x,t)\right]-kx^{m-1}\frac{\partial}{\partial t}u(x,t)=-Q(x,t).$$

They correspond for positive integers m to heat conduction with spherical symmetry in an m-dimensional medium. A mixed Dirichlet-Neumann problem for u is to be solved. Let R be a region in the (ξ,τ)-plane bounded by a segment C_1 given by $\tau=\text{constant}=t$, $\alpha\leqq\xi\leqq\beta$, and a curve C_2, for which $\tau<t$. A solution u of (1) is to be found on C_1 from the condition that

$$(2)\qquad u(\xi,\tau)+\beta(\xi,\tau)\frac{\partial}{\partial\xi}u(\xi,\tau)=\xi^{m-1}f(\xi,\tau)$$

on C_2. Using Green's identity the determination of u is reduced to that of a Riemann function $v(x,t,\xi,\tau)$, which satisfies the homogeneous differential equation adjoint to (1) in R, satisfies a homogeneous condition similar to (2) on C_2, and the condition $k\xi^{m-1}v(x,t,\xi,t)=\delta(x-\xi)$, where δ is the Dirac function.

The question of the existence of such a function v and the verification of the boundary condition for u are left open. It is pointed out, however, that a necessary condition for the existence of v is that $\beta=0$ wherever $\cos(n,\xi)=0$ on C_2. Some attention is given to the case of a free moving boundary in the medium, which occurs in connection with freezing and melting of solids. *F. John* (New York, N. Y.).

***Vernotte, Pierre. Thermocinétique.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. **224**, xxi+459 pp. (1949).

This book is a series of meditations and reflections on the movement of heat. The author's personal criticisms of the solutions of various problems are interesting and thought-provoking. Starting with a discussion of the various means of heat conduction, Fourier's law is stated, and then the heat equation is derived. Following this are three chapters that should prove the most interesting to mathematicians. First, problems on the movement of heat in permanent regions are considered, that is, problems associated with Laplace's equation. Then boundary value problems for the

heat equation in finite regions are solved using Fourier developments and theta functions. Conduction of heat in infinite regions is treated by use of the fundamental solution of the parabolic equation. While the mathematical treatments of the problems are not new, the author's discussion of their practical value in measurement of the temperature over very short and long periods of time are interesting and instructive. The next part of the book is devoted to the discussion of experiments and methods for measuring the various physical constants that enter the heat equation. The last part of the book reproduces some twenty-seven research papers of the author on various aspects of heat conduction. Throughout the book the boundary value problems are in general associated with walls or bars, so that essentially all problems are solved using rectangular coordinates. This relieves the author from any long and involved mathematical manipulation, and gives him more time for his main purpose, which is discussing the physical aspects of heat conduction. *F. G. Dressel* (Durham, N. C.).

Casci, Corrado. Sulla distribuzione delle temperature in regime permanente di un anello in ambienti a temperatura diversa. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 219–224 (1950).

A ring placed in two surrounding media A and B, which are separated by a diametral plane of the ring, is assumed to be in the permanent state regarding its temperature. Assuming medium A at constant temperature θ_A, and region B at constant temperature θ_B, the author finds the temperature θ interior to the ring. Analytically the problem is the solution θ of the differential equation:

$$d^2\theta/ds^2+\alpha(\theta-\theta^*)=0, \qquad 0\leqq s\leqq 2\pi r,$$

where $\theta^*=\theta_A$ for $0<s<\pi r$, $\theta^*=\theta_B$ for $\pi r<s<2\pi r$, and r and α are constants. *F. G. Dressel* (Durham, N. C.).

Casci, Corrado. Sulla distribuzione della temperatura in un anello rotante in ambienti a temperatura diversa. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 297–303 (1950).

A ring imbedded in two surrounding media A and B, which are separated by a diametral plane of the ring, is assumed to be rotating with a constant angular velocity ω. The problem of the paper is to find the distribution of temperature θ of the ring, given that region A is at constant temperature θ_A and region B is maintained at constant temperature θ_B; the initial temperature θ_0 of the ring is also assumed known. The author finds θ by the method of separation of variables, where θ satisfies the conditions

$$\partial^2\theta/\partial s^2+\alpha(\theta-\theta^*)=\beta\partial\theta/\partial t, \qquad \theta(s,0)=\theta_0(s).$$

Here $\theta^*=\theta_A$ for $0<s'<\pi r$, $\theta^*=\theta_B$ for $\pi r<s'<2\pi r$, and s' is related to s and t by the relation $s'=s+\omega rt$. The coefficients α and β are assumed constant and r is the radius of the ring. *F. G. Dressel* (Durham, N. C.).

Kataoka, Hidekichi. Linear conduction of heat in a compound rod. II. Trans. Soc. Mech. Engrs. Japan **14**, no. 46, 168–170 (1948). (Japanese)

Kataoka, Hidekichi. Linear conduction of heat in a compound rod. III. Trans. Soc. Mech. Engrs. Japan **14**, no. 47, part 2, 29–35 (1948). (Japanese)

According to a review in Appl. Mech. Rev., the author solves the equation of linear heat conduction for a rod composed of two rods of different materials, with heat supplied at one end and radiated from the other.

Difference Equations, Special Functional Equations

***Picard, Émile. Leçons sur quelques équations fonctionnelles avec des applications à divers problèmes d'analyse et de physique mathématique. Rédigées par Eugène Blanc.** Gauthier-Villars, Paris, 1950. iii+187 pp. 400 francs.

"Nouveau tirage" of a work first published in 1928.

Töpfer, Hans. Über die meromorphen Lösungen der Differenzengleichung $F(z+1)=\dfrac{R_1(z)F(z)+R_2(z)}{F(z)+R_3(z)}$ mit rationalen $R_\nu(z)$. Math. Z. **52**, 436–471 (1949).

The totality of all meromorphic solutions of the equation (1) of the title is found, under the hypothesis that if

$$R(z)=[R_2(z)-R_1(z)R_3(z)][R_1(z-1)+R_3(z)]^{-1}\times[R_1(z)+R_3(z+1)]^{-1},$$

then $R(\infty)\neq-1$. The first step in the method consists in transforming (1) by a suitable substitution

$$F(z)=A_1(z)f(z)+A_2(z),$$

with rational $A_j(z)$, into one of the two forms

$$f(z)f(z+1)=S(z), \tag{2}$$

or

$$f(z+1)=R(z)/[1+f(z)]. \tag{3}$$

Equation (2) is solved at once, and the bulk of the paper is devoted to the problem of finding a meromorphic solution of (3). This is done by an iterative process leading to a sequence of functions which are modifications of the convergents of the continued fraction associated with (3). This sequence is investigated by techniques from the theories of normal families, and of attractive fixed points in iteration. The investigation is formulated in terms of the more general equation $f(z+1)=H(z,f(z))$, where $H(v,w)$ is a given function meromorphic for all finite v and w. However, the convergence discussions require that H satisfy additional conditions which in the general case are of doubtful verifiability, and it is only in the special cases (3) and (4) $f(z+1)=[af(z)+b]/z$ that these conditions are shown to be satisfied. The results for (1) are applied to secure, for linear homogeneous difference equations of the second order, solutions in a form similar to the Thiele continued fraction solutions, convergent in some cases where the corresponding continued fractions are divergent. *W. Strodt.*

Myškis, A. On the solutions of linear homogeneous differential equations of the first order of unstable type with a retarded argument. Doklady Akad. Nauk SSSR (N.S.) **70**, 953–956 (1950). (Russian)

Under consideration is the equation

$$y'(x)-M(x)y(x-\Delta(x))=0 \tag{1}$$

($M(x)\geqq 0, \Delta(x)\geqq 0; A\leqq x<B$; finite A, B). Let E_0 consist of A and of the set of values $x-\Delta(x)\leqq A$. On E_0 there is assigned a function $\varphi(x)$; $M(x)$, $\Delta(x)$, $\varphi(x)$ are continuous; $y(x)$ is to satisfy (1) so that $y(x-\Delta(x))\equiv\varphi(x-\Delta(x))$, wherever $x-\Delta(x)<A$, and $y(A)=\varphi(A)$; $\delta_0=\inf\Delta(x)$, $\Delta_0=\sup\Delta(x)$, $m_0=\inf M(x)$, $M_0=\sup M(x)$, on $[A,B)$; $\varphi_0=\inf\varphi(x)$, $\Phi_0=\sup\varphi(x)$, on E_0; with a in $[A,B]$, let $\gamma(a)$ be the upper bound of those x in (a,B), for which $x-\Delta(x)<a$. It has been proved previously [Myškis, Uspehi Matem. Nauk (N.S.) **4**, no. 5(33), 99–141 (1949); these Rev. **11**, 365] that there is a unique solution on $[A,B)$. Some of the results are as follows. If $\varphi_0\geqq 0$, $y(x)$ is monotone non-

decreasing; if $\varphi_0>0$ and $m_0>0$, then $y(x)$ is monotone increasing. If $A\leqq a=\gamma(a)<B$, then: (a) if $y(a)=0$, $y(x)\equiv 0$ for $x\geqq a$; (b) if $y(a)>0$, $y(x)$ is monotone nondecreasing for $x\geqq a$; (c) if $y(a)>0$ and $m_0>0$, $y(x)$ is monotone increasing. Let $A\leqq a<\gamma(a)<B$; (a) if $y(x)\geqq 0$ on $[a, \gamma(a)]$, $y(x)$ is monotone nondecreasing for $x\geqq\gamma(a)$; (b) if $y(x)>0$ on $[a, \gamma(a)]$ and $m_0>0$, y is monotone increasing for $x\geqq\gamma(a)$.

Given equation (2) $y'(x)-M_1(x)y(x-\Delta_1(x))=0$ $(A\leqq x<B)$, with the initial function $\varphi_1(x)$ and solution $y_1(x)$; (a) if $0\leqq M(x)\leqq M_1(x)$, $0\leqq\varphi(x)\leqq\varphi_1(x)$, $\Delta(x)=\Delta_1(x)$, then $y_1(x)\geqq y(x)$; (b) if $0\leqq M(x)\leqq M_1(x)$, $\Delta_1(x)\leqq\Delta(x)$ and $\varphi(x)$ $(\varphi_1(x))$ is given also on $E_{1,0}$ (E_0), while therein $0\leqq\varphi(x)\leqq\varphi_1(x)$, and $\varphi(x)$ $(\varphi_1(x))$ is monotone nondecreasing on $E_0+E_{1,0}$, then $y_1(x)\geqq y(x)$. Write $a_0=A$, $a_{k+1}=\gamma(a_k)$ $(k=0, 1, \cdots)$; let T be the upper bound of the right endpoints of those $[a_k, a_{k+1}]$ on which $y(x)$ has zeros. Suppose $T<B$; (a) if $T=\lim a_k$, then $y(x)\equiv 0$ $(x\geqq T)$; (b) if $T=a_k$, $a_{k+1}<B$, then $|y(x)|$ is monotone increasing $(x\geqq a_{k+1})$. Moreover, $|y(x)|\leqq\sup_{x_0}|\varphi(x)|\cdot(M_0\Delta_0)^\nu$ $(A\leqq x<T)$, where $\nu=\Delta_0^{-1}(x-A)$. *W. J. Trjitzinsky* (Urbana, Ill.).

***Minorsky, N. Self excited oscillations in systems possessing retarded actions.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, pp. 43–51.

The author gives a brief discussion of the types of applied problems that lead to differential-difference equations. After a treatment of the linear equation, he proceeds to the nonlinear equation and presents a heuristic method for finding the amplitude and frequency of a periodic solution.

R. Bellman (Stanford University, Calif.).

Fujii, Sumiji. Time-lag vibration. Rep. Inst. Sci. Tech. Univ. Tokyo 1, 80–83 (1947). (Japanese)

The author studies special cases of $\psi(t)=\Phi\{\psi(t-\tau)\}$, where Φ is an operator, e.g., $\psi(t)=\alpha\psi(t-\tau)$ and $\theta'(t)+\theta(t-\tau)=0$.

Integral Equations

***Muskhelishvili, N. I. Singular Integral Equations. Boundary Problems of Function Theory and Their Application to Mathematical Physics. Translated by J. R. M. Radok and W. G. Woolnough.** Department of Supply and Development, Aeronautical Research Laboratories, Melbourne, Australia, Translation No. 12, 1949. xi+404 pp.

The Russian original was published by OGIZ, Moscow-Leningrad, 1946; these Rev. 8, 586.

Chang, Shih-Hsun. On the distribution of the characteristic values and singular values of linear integral equations. Trans. Amer. Math. Soc. 67, 351–367 (1949).

Defining a characteristic value of a real L^2 kernel $K(x, y)$ as a number μ for which the equation

$$\varphi(x)=\mu\int K(x, y)\varphi(y)dy$$

has a nonnull L^2 solution $\varphi(x)$, and a singular value of the same kernel to be a number λ for which the pair of equations

$$\varphi(x)=\lambda\int K(x, y)\psi(y)dy,\quad \psi(x)=\lambda\int K(y, x)\varphi(y)dy$$

has nonnull L^2 solutions $\varphi(x)$, $\psi(x)$, the author proves the following results, and gives a number of applications of them. (i) If (μ_n), (λ_n) are the sequences of characteristic values and singular values, respectively, of the kernel $K(x, y)$ and $\tau>0$, the convergence of the series $\sum|\lambda_n|^{-\tau}$ implies the convergence of $\sum|\mu_n|^{-\tau}$. (ii) If $D(\lambda)=\sum_{n=0}^{\infty}(-1)^nC_n\lambda^n$ is the Fredholm determinant of the kernel $\int K(x, s)K(y, s)ds$ and $0<\rho<2$, the series $\sum|\lambda_n|^{-\rho}$ is convergent if and only if $\sum|C_n|^{\rho/2n}$ is convergent. (iii) If

$$K_1K_2(x, y)=\int K_1(x, s)K_2(s, y)ds$$

and the m-fold product is defined inductively in the obvious way, a necessary and sufficient condition for the L^2 kernel $K(x, y)$ to be expressible in the form $K_1K_2\cdots K_m(x, y)$, where each K_i is an L^2 kernel, is that $\sum|\lambda_n|^{-2/m}$ be convergent. In obtaining these results, the author makes considerable use of the theory of entire functions. A direct proof of (i) has now been given by Weyl [Proc. Nat. Acad. Sci. U. S. A. 35, 408–411 (1949); these Rev. 11, 37].

F. Smithies (Cambridge, England).

Fuchs, K. Perturbation theory in neutron multiplication problems. Proc. Phys. Soc. Sect. A. 62, 791–799 (1949).

This paper develops a perturbation theory for the solution of the integral equation

$$(1)\quad n(r)=\int\beta(r')K(r, r')n(r')dV'+\int v^{-1}q(r')K(r, r')dV',$$

where $\beta(r)$ and $q(r)$ are some known functions of $r=(x, y, z)$, $K(r, r')$ is a function symmetrical in the variables r and r', v is a constant and dV' is the element of volume $dx'dy'dz'$. It is known that the homogeneous equation

$$f(r)=\lambda\int\beta(r)K(r, r')f(r')dr'$$

associated with (1) has solutions $f_i(r)$ for an infinite number of eigenvalues λ_i and that these solutions form a complete set of functions orthogonal to the set $\beta(r)f_i(r)$. Now if β and K are subject to small variations $\Delta\beta(r)$ and $\Delta K(r, r')$, then the corresponding change $\Delta\lambda_i$ in the eigenvalue λ_i is given by

$$(2)\quad \lambda_i^{-1}\Delta\lambda_i+\int\Delta\beta(r)[f_i(r)]^2dV+\int\int f_i(r)\beta(r)\Delta K(r, r')\beta(r')f_i(r')dVdV'=0,$$

if the functions f_i are further assumed to be normalized. The solution of the nonhomogeneous equation (1) can be found by expanding $n(r)$ in terms of $f_i(r)$ in the form

$$n(r)=\sum[n]_if_i(r);\quad [n]_i=\int\beta(r)n(r)f_i(r)dV.$$

Substituting this in (1), we find

$$(\lambda_i-1)[n]_i=v^{-1}[q/\beta]_i=v^{-1}\int q(r)f_i(r)dV.$$

In this manner the author obtains the solution

$$(3)\quad vn(r)=\sum_i(\lambda_i-1)^{-1}[q/\beta]_if_i(r).$$

The author discusses various applications of (2) and (3).

S. Chandrasekhar (Williams Bay, Wis.).

Okamura, Hirosi. Sur certaines équations de Volterra singulières. Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. 22, 429–453 (1939).

The equations $(I_\pm)$: $f(x)\pm\int_0^x k(x, y)f(y)dy=g(x)$ are studied, where $k(x, y)=m(x)+l(x, y)$, $m(x)\geqq 0$, $\int_0^\infty m(x)dx=\infty$

but $\int_\epsilon m(x)dx < \infty$ for every $\epsilon > 0$, and $l(x, y)$, roughly speaking, is smooth and small compared with $m(x)$ where $x = y$ and x is small. Necessary conditions on $g(x)$ for the existence of an integrable solution $f(x)$ are found; if they are satisfied, (I_+) has a unique solution, (I_-) a linear family of solutions $f_1(x) + cf_2(x)$ where $f_2(x)$ satisfies the homogeneous form of (I_-). The integrals are taken in the Lebesgue sense or in the Lebesgue-Cauchy sense at the lower limit. The results can be applied to an equation of the first kind (II): $\int_0^x K(x, y)F(y)dy = G(x)$ with an analytic kernel $K(x, y) = ax^n + by^n + \sum_{p+q>n} c_{pq}x^p y^q$, $a \neq 0$ or $b \neq 0$, as differentiation of (II) with respect to x leads to an equation of type (I_+). *G. E. H. Reuter* (Manchester).

Kostitzin, Vladimir-A. Sur l'équation intégrale du cycle fermé. C. R. Acad. Sci. Paris **230**, 811–813 (1950).

This note is devoted to a brief discussion of some of the calculational problems which arise from the solution of integral equations of the type

$$\phi(x) = f(x) + \int_0^t K(x-t)\phi(t)dt.$$

Here $f(x)$ and $K(x)$ are known and $\phi(x)$ is sought. *A. E. Heins* (Pittsburgh, Pa.).

Miles, John W. On certain integral equations in diffraction theory. J. Math. Physics **28**, 223–226 (1950).

The problem of the diffraction of a plane wave by a ribbon or a slit of infinite length in an infinite screen is formulated as an integral equation which may be solved by Mathieu functions. Some identities involving Mathieu functions are derived from these integral equations. *A. E. Heins* (Pittsburgh, Pa.).

Block, Henry David. Explicit solution of certain singular integral equations. Iowa State Coll. J. Sci. **24**, 14–16 (1949).

Typical of the class of integral equations are

$$f(x) = g(x) - \lambda \int_0^\infty |x-y| e^{-k|x-y|} g(y)dy$$

and

$$f(x) = g(x) - \lambda \int_{-\infty}^\infty |x-y| e^{-k|x-y|} g(y)dy,$$

where $g(x)$ is sought and $f(x)$ is known. These equations are solved by forming a differential equation for $g(x)$. Some remarks are made on the types of differential operators required to annihilate the integral. *A. E. Heins.*

Functional Analysis

Sebastião e Silva, J. Analytic functions and functional analysis. Portugaliae Math. **9**, 1–130 (1950). (Portuguese. French summary)

This paper appears to be inspired in part by the work of Pincherle and Fantappiè, but it is written more in the recent style of functional analysis in which abstract spaces, particularly Banach spaces, play a central role. A considerable portion of the paper is given over to the exposition of background material and known results. There is an eight page summary in French. Let C be a closed subset of the extended complex plane Ω, both C and $\Omega - C$ being nonvoid. An open set of Ω is called an open neighborhood of C if it contains C, is bounded if C is bounded, and contains no component not intersecting C. Two functions f_1, f_2, each defined and holomorphic in open neighborhoods, D_1, D_2 respectively, of C, are called equivalent relative to C if there is an open neighborhood D of C in which $f_1(z) = f_2(z)$, D lying in the intersection of D_1 and D_2. An equivalence class of such functions is called an analytic function associated with C, and the set of elements consisting of all such classes is denoted by $\mathfrak{F}[C]$ if C is bounded. If the point ∞ belongs to C the additional requirement is made that all the functions vanish at ∞. Then $\mathfrak{F}[C]$ is a vector space; it is a limit space of type (L) in the sense of Fréchet, convergence of elements meaning uniform convergence on some open neighborhood of C.

The first fundamental problem of the paper is this: if S is an abstract vector space of type (L), what is the general form of a continuous linear mapping F of $\mathfrak{F}[C]$ into S? If S is a complex Banach space the answer is that F has the representation $F[\varphi] = (2\pi i)^{-1}\int_\Gamma \varphi(\lambda)\mathbf{f}(\lambda)d\lambda$, Γ being a suitable set of contours, and $\mathbf{f}(\lambda)$ being a function with values in S, holomorphic on $\Omega - C$ (with $\mathbf{f}(\infty) = 0$ if C is bounded). The latter function is called the indicatrix of F, and is given by $F[\varphi]$ when $\varphi(z) = (\lambda - z)^{-1}$. The second problem is prefaced by the observation that if C is bounded, $\mathfrak{F}[C]$ is a commutative Banach algebra with a unit. If $\mathbf{A}$ is an abstract Banach algebra with unit, what can be said about continuous, unit-preserving homomorphisms of $\mathfrak{F}[C]$ into $\mathbf{A}$? If $\mathbf{a} \in \mathbf{A}$, and if we require further that the homomorphism F have the property $F[\varphi] = \mathbf{a}$ when $\varphi(z) = z$, a necessary and sufficient condition that such an F exist is that the spectrum of $\mathbf{a}$ lie in C. In the latter event, F is uniquely given by $F[\varphi] = (2\pi i)^{-1}\int_\Gamma \varphi(\lambda)(\lambda - \mathbf{a})^{-1}d\lambda$. The paper continues with a discussion of the implications of the notation $F[\varphi] = \varphi(\mathbf{a})$. Here connection is made with recent work on operational calculus by Dunford, Hille, Lorch, Taylor, and others. The author lists Hille's Colloquium Lectures [1944] in his bibliography, but apparently had not seen the book form of these lectures [Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloquium Publ., v. 31, New York, 1948; these Rev. **9**, 594]. Generalizations and applications are considered. *A. E. Taylor* (Los Angeles, Calif.).

Dieudonné, J. Natural homomorphisms in Banach spaces. Proc. Amer. Math. Soc. **1**, 54–59 (1950).

If G is a linear subspace of a Banach space E, G^* and E^* their respective duals, there is a natural homomorphism of E^* onto G^*. M. E. Munroe [Bull. Amer. Math. Soc. **54**, 776–781 (1948); these Rev. **10**, 128] showed that this homomorphism is open in the weak and weak* topologies. The present paper points out that these results are particular cases of a theorem of the author [Ann. Sci. École Norm. Sup. (3) **59**, 107–139 (1942); these Rev. **6**, 178] and proves that T is also open in the bounded weak and bounded weak* topologies. A theorem characterises these topologies by fundamental systems of neighbourhoods. *J. L. B. Cooper* (London).

Dixmier, Jacques. Applications ♮ dans les anneaux d'opérateurs. C. R. Acad. Sci. Paris **230**, 607–608 (1950).

In earlier work [Ann. Sci. École Norm. Sup. (3) **66**, 209–261 (1949); these Rev. **11**, 370] the author has shown that for a ring of operators R of finite class there is a unique mapping $A \to A^\natural$ of the ring on its center having certain natural algebraic properties. This result applied to the

special case in which R is a factor gives the existence and uniqueness of the von Neumann-Murray relative trace. The present note announces an extension of these results in which the restriction that R is of finite class is removed. Here the fundamental mapping $\phi: A \to A^{\phi}$ is defined in general only on an ideal m in R. It is maximal among all "positive" "central" mappings of ideals m of R into the center of R which have a certain continuity property. It is not unique but is determined (roughly speaking) up to multiplication by a positive number in each component of the von Neumann decomposition of the ring into factors. (Actually the decomposition of a ring into factors is not discussed and the uniqueness is formulated in another manner.)

In a second paragraph "traces" on R are discussed. For any particular ϕ the traces defined on its domain are in one-to-one correspondence with the positive linear functionals defined on its range. The final theorem announced asserts that there is a one-to-one correspondence between traces on R and the "weight functions" defined on its projections which is analogous to the classical correspondence between integrals and measures. For more precise statements of results and further details the reader must be referred to the highly condensed note itself.

G. W. Mackey (Cambridge, Mass.).

Dvoretzky, A., and Rogers, C. A. Absolute and unconditional convergence in normed linear spaces. Proc. Nat. Acad. Sci. U. S. A. **36**, 192–197 (1950).

In this note the authors verify a conjecture open for about twenty years. Theorem 1. Absolute convergence is equivalent to unconditional convergence of series of points of a Banach space if and only if the space is finite-dimensional. This result follows from their theorem 2. If B is an infinite-dimensional Banach space and $\sum_\nu c_\nu$ is any convergent series of positive terms, then there exists an unconditionally convergent series $\sum_\nu x_\nu$ of points of B such that $\|x_\nu\|^2 = c_\nu$ for $\nu = 1, 2, \cdots$. The proof depends on a geometric result in Euclidean n-space. Lemma 1. Let C be a convex body with center at the origin; then there are n points $A_1, \cdots, A_n$ on the boundary of C such that for $\lambda_1, \cdots, \lambda_r$ real, $1 \leqq r \leqq n$, the point $\lambda_1 A_1 + \cdots + \lambda_r A_r$ is in λC where $\lambda^2 = [2 + r(r-1)/n](\lambda_1^2 + \cdots + \lambda_r^2)$. Applying this with $n = r(r-1)$ gives lemma 2. Let B be a Banach space of infinite dimension; then there exist $x_1, \cdots, x_r \epsilon B$ such that $\|x_\nu\|^2 = c_\nu$ for $\nu = 1, \cdots, r$ and $\|\sum' x_\nu\|^2 \leqq 3\sum' c_\nu$, where, on both sides, $\sum'$ is extended over the same subset of the integers $1, \cdots, r$. A series satisfying theorem 2 is constructed by applying lemma 2 to each block of terms $n_r < \nu \leqq n_{r+1}$, where the n_r, $n_{r+1} > n_r$, are chosen so that $\sum_{r=1}^{\infty} (\sum^* c_\nu)^{\frac{1}{2}} < \infty$, where $\sum^*$ extends over $n_r + 1 \leqq \nu \leqq n_{r+1}$.

M. M. Day (Urbana, Ill.).

Al'tman, M. Š. On bases in Hilbert space. Doklady Akad. Nauk SSSR (N.S.) **69**, 483–485 (1949). (Russian)

Continuing earlier work of the author [same Doklady **67**, 413–416 (1948); these Rev. **11**, 114] and of K. I. Babenko [ibid. **62**, 157–160 (1948); these Rev. **10**, 249] this note gives an example of a Schauder basis $\{f_i\}$ in Hilbert space H which is neither a Hilbert basis nor a Bessel basis; i.e., letting $\{g_i\}$ be the biorthogonal basis to $\{f_i\}$, there exist numbers a_i such that $\sum |a_i|^2 < \infty$ but no x in H satisfies $(x, g_i) = a_i$ for all i, and there exists $x \epsilon H$ such that $\sum |(x, g_i)|^2 = \infty$.

M. M. Day (Urbana, Ill.).

Grinblyum, M. M. On the representation of a space of type B in the form of a direct sum of subspaces. Doklady Akad. Nauk SSSR (N.S.) **70**, 749–752 (1950). (Russian)

This paper discusses decomposition of a Banach space into a direct sum of two or countably many closed linear subspaces. For example, let $\{P_i\}$ be a sequence of closed linear subspaces of a Banach space E; for each subset A of the set I of integers, let P_A (P^A) be the smallest closed linear subspace of E containing all P_i, $i \epsilon A$ (all P_i, $i \bar{\epsilon} A$); let S be the set of points of norm one in E. Theorem: the vector sum of all the spaces P_i is closed in P_I if and only if there exists $\alpha > 0$ such that for each n the distance $d(S \cap P_{1,2,\cdots,n}, P^{1,2,\cdots,n}) \geqq \alpha$. *M. M. Day* (Urbana, Ill.).

Grinblyum, M. M. Spectral function in a Banach space. Doklady Akad. Nauk SSSR (N.S.) **70**, 941–944 (1950). (Russian)

This paper considers one-parameter families of projections in a reflexive Banach space E. For each real λ, let P_λ and P^λ be closed linear subspaces of E such that E is the direct sum of P_λ and P^λ while P_λ increases toward E and P^λ decreases toward 0 as $\lambda \to \infty$. Say that $\{P_\lambda; P^\lambda\}$ is an α-system if $d(S \cap P_\lambda, P^\lambda) \geqq \alpha > 0$ for all λ with $P_\lambda \neq 0$. For an α-system it is shown that right and left hand limits of P_λ and P^λ exist at each λ. A certain finitely-additive function of intervals defined from an α-system is called a spectral function if the system satisfies a stronger disjointness property. *M. M. Day* (Urbana, Ill.).

Atkinson, F. V. Symmetric linear operators on a Banach space. Monatsh. Math. **53**, 278–297 (1949).

The author generalises some results about symmetric operators in Hilbert space to a general complex Banach space. In place of symmetric operators he considers operators U from a complex Banach space into itself having one of the following properties: (A) $\|Uf\|^2 \leqq \|f\| \cdot \|U^2 f\|$ for every element f of the space; (B) every polynomial in U with constant complex coefficients has property (A). He further defines f as orthogonal to g if $\|f + \lambda g\| \geqq \|f\|$ for every scalar λ. The results he proves include the following. If U is a bounded linear operator satisfying (B), any two eigen-functions corresponding to different eigen-values are mutually orthogonal. If the bounded linear operator U satisfies (B) and has a discrete spectrum $(\lambda_1, \lambda_2, \cdots)$ with no limit point but the origin, then there is a sequence of mutually orthogonal projectors $P_1, P_2, \cdots$ with bound 1 such that $U = \sum_{r=1}^{\infty} \lambda_r P_r$. [Note: There are a number of misprints, of which the more important are as follows. P. 285: the expression at the start of line 7 should read "$\|(\lambda_n I - U)^{-1}\|$." Pp. 286 (line 24) and 290 (line 8): "m" has been replaced by "n." P. 290, line 11: "$\|\phi\|$" should be omitted from the end of the formula.] *A. F. Ruston.*

Fan, Ky. Partially ordered additive groups of continuous functions. Ann. of Math. (2) **51**, 409–427 (1950).

L'auteur se propose de caractériser le groupe ordonné $C(\Omega)$ des fonctions continues dans un espace compact Ω, et certains de ses sous-groupes, par des propriétés ne faisant intervenir que la structure de groupe ordonné, en excluant toute hypothèse sur le caractère réticulé ("lattice") des groupes envisagés. Il commence (théorème 1) par caractériser ainsi tous les sous-groupes G de $C(\Omega)$ qui contiennent toutes les constantes et séparent deux points quelconques de Ω. Pour cela, il suppose que G contient un groupe isomorphe au groupe ordonné R des réels, et que les éléments positifs de R sont "archimédiens" dans G, c'est-à-dire que

si f est un tel élément et g quelconque dans G, il existe un entier n tel que $g \leqq nf$; enfin, il suppose que si f, g sont deux éléments de G tels que $nf+g \geqq 0$ pour tout entier $n>0$, alors $f \geqq 0$. Pour déduire de ces trois conditions la caractérisation en question, l'auteur définit, comme d'ordinaire, les points de Ω comme certains sous-groupes de G possédant une propriété de "maximalité": ici, il s'agit des sous-groupes maximaux parmi les sous-groupes $\neq G$ et "convexes" au sens suivant: si h_1, h_2 sont deux éléments d'un tel sous-groupe H, et si $h_1 \leqq f \leqq h_2$, alors $f \in H$. Mais l'espace des sous-groupes convexes maximaux d'un groupe G (satisfaisant aux conditions précédentes) est en général trop grand: si on part d'un groupe G de la forme $C(\Omega)$, on ne retrouve pas ainsi l'espace compact Ω, mais un espace le contenant. Pour retrouver Ω, il faut considérer, non plus les sous-groupes maximaux convexes, mais les sous-groupes maximaux "singuliers," un sous-groupe singulier de G étant défini comme un sous-groupe de G qui est filtrant et dont tous les éléments sont non-archimédiens. L'auteur montre qu'on peut obtenir ainsi Ω, non seulement lorsque $G=C(\Omega)$, mais encore lorsque G est un sous-groupe de $C(\Omega)$ contenant les constantes, séparant les points de Ω, et tel en outre que le sous-groupe de G formé des fonctions s'annulant en un point quelconque de Ω soit filtrant. Il donne une caractérisation abstraite de ces groupes par une quatrième condition reliant les notions de sous-groupe convexe maximal et de sous-groupe singulier maximal; si on suppose enfin qu'en un certain sens, G ne peut être plongé dans un sous-groupe plus grand vérifiant les mêmes quatre conditions, alors G est isomorphe à un $C(\Omega)$.

J. Dieudonné (Nancy).

Fan, Ky. On a theorem of Weyl concerning eigenvalues of linear transformations. II. Proc. Nat. Acad. Sci. U. S. A. **36**, 31–35 (1950).

The author proves the following results on linear operators in an n-dimensional unitary space. (i) Let A be a linear operator, and let β, γ be nonnegative and have sum 1. Let the eigenvalues of A and of $H=\beta A^*A+\gamma AA^*$ be (λ_i) and (ξ_i), where

$$|\lambda_1| \geqq \cdots \geqq |\lambda_n|, \quad \xi_1 \geqq \cdots \geqq \xi_n.$$

If $\omega(t)$ is a nondecreasing function on $t \geqq 0$ such that $\omega(e^\tau)$ is a convex function of τ, then

$$\sum_{i=1}^{q} \omega(|\lambda_i|^2) \leqq \sum_{i=1}^{q} \omega(\xi_i), \qquad 1 \leqq q \leqq n.$$

The case $\beta=1$ of this was proved by Weyl [same Proc. **35**, 408–411 (1949); these Rev. **11**, 37]. (ii) If A has the eigenvalues (λ_i) and $\frac{1}{2}(A+A^*)$ has the eigenvalues (ρ_i), where

$$\Re\lambda_1 \geqq \cdots \geqq \Re\lambda_n, \quad \rho_1 \geqq \cdots \geqq \rho_n,$$

then

$$\sum_{i=1}^{q} \Omega(\Re\lambda_i) \leqq \sum_{i=1}^{q} \Omega(\rho_i), \qquad 1 \leqq q \leqq n.$$

for any non-decreasing convex function $\Omega(t)$. Extensions to completely continuous operators in Hilbert space are indicated. *F. Smithies* (Cambridge, England).

Polya, George. Remark on Weyl's note "Inequalities between the two kinds of eigenvalues of a linear transformation." Proc. Nat. Acad. Sci. U. S. A. **36**, 49–51 (1950).

The author shows that a result of Weyl [same Proc. **35**, 408–411 (1949); these Rev. **11**, 37] can be proved by using the following lemma. If $a_1, \cdots, a_m$ and $b_1, \cdots, b_m$ are sets of real numbers such that $b_1 \geqq b_2 \geqq \cdots \geqq b_m$ and $b_1+\cdots+b_q \leqq a_1+\cdots+a_q$, $1 \leqq q \leqq m$, then

$$\omega(b_1)+\cdots+\omega(b_m) \leqq \omega(a_1)+\cdots+\omega(a_m)$$

for any convex increasing function $\omega(x)$. *F. Smithies.*

Nemyckiĭ, V. V. Topological problems of the theory of dynamical systems. Uspehi Matem. Nauk (N.S.) **4**, no. 6(34), 91–153 (1949). (Russian)

This is an expository article on regular curve families [Whitney], universal dynamical systems [Bebutov], completely unstable dynamical systems [Nemyckiĭ], central motions [Maĭer], and the structure of orbit-closures [Hil'mi and others]. Some twenty unsolved problems are listed. The paper also contains historical remarks; those on the Russian school of topological dynamics are particularly welcome. This article is complementary to the recent book by the author and Stepanov [Qualitative Theory of Differential Equations, OGIZ, Moscow-Leningrad, 1947; these Rev. **10**, 612]. The most important part of the paper is the proof (previously inaccessible) of Bebutov's theorem: every dynamical system on a compact metric space with at most one fixed point can be topologically imbedded in the dynamical system U of all continuous functions on the line to the line, U being provided with the compact-open topology and the flow in U being defined by translations.

W. H. Gottschalk (Philadelphia, Pa.).

Calculus of Variations

Viola, Tullio. Sulla ricerca delle estremanti d'un integrale in forma ordinaria, alla frontiera d'un campo dello spazio funzionale lagrangiano del prim'ordine. Ann. Scuola Norm. Super. Pisa (3) **1** (1947), 101–160 (1949).

This paper considers the problem of extremizing integrals of the form $\int_a^b f(x, y, y')dx$ depending on plane curves $y=y(x)$ subject to the following restrictions: (i) $y(x)$ and $y'(x)$ are continuous, and there is a nonnegative integrable function $L(x)$ such that $|y'(x_2)-y'(x_1)| \leqq \int_{x_1}^{x_2} L(x)dx$ for every interval (x_1, x_2); (ii) the values of $y(x)$ and $y'(x)$ are prescribed at certain points of $[a, b]$. This is an extension of previous work of the author, for the case $f=(1+y'^2)^{\frac{1}{2}}$, $L(x)=$constant>0 [see Rend. Circ. Mat. Palermo **62**, 105–136 (1939)]. Sufficient conditions are given for a minimum and for a maximum of the integral in such classes of admissible curves. Naturally the methods and results are quite different from those of the classical calculus of variations.

L. M. Graves (Chicago, Ill.).

Sibagaki, Wasawo. Concerning abnormalities of admissible arcs in the problem of Bolza in the calculus of variations. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **3**, 81–101 (1944). (Japanese)

The present paper is devoted to a systematic study of the concept of abnormality and its relation to the multiplier rule. The paper is based largely on the works of G. A. Bliss, initially given in a set of lecture notes and now found in the closing chapters of his book [Lectures on the Calculus of Variations, University of Chicago Press, Chicago, Ill., 1946; these Rev. **8**, 212]. The results obtained follow readily from those given by Bliss. The main theorem consists in showing that an admissible arc has abnormality q if and only if it satisfies the multiplier rule with a set of q linearly independent sets of multipliers, each having $\lambda_0=0$.

M. R. Hestenes.

Dedecker, Paul. Sur une méthode de Bateman dans le problème inverse du calcul des variations. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 774–792 (1949).

This paper is concerned with the question of deriving a given set of partial differential equations from a variational principle. The author points out the incompleteness of present knowledge, referring to available results and emphasizing the fact that there are really two questions involved: (a) the identity of a set of partial differential expressions with a set of variational derivatives; (b) the equivalence of the two sets of partial differential equations obtained by equating these to zero. He describes how H. Bateman [Physical Rev. (2) **38**, 815–819 (1931)], by introducing additional variables z^σ, obtained the partial differential equations $F_\sigma=0$ from the variational principle $\delta\int z^\sigma F_\sigma dx^1\cdots dx^p=0$. He discusses the geometrization of Bateman's formulae and their tensor character under transformations, and considers some particular cases.

J. L. Synge (Dublin).

Giannopoulos, Alex. I. The Du Bois-Reymond lemma. Bull. Soc. Math. Grèce **24**, 26–27 (1949). (Greek)

Mathematical Statistics

Gini, Corrado. Considerazioni sulle probabilità a posteriori e applicazioni al rapporto dei sessi nelle nascite umane. Metron **15**, 133–171 (1949).

Gumbel, E. J. Probability tables for the range. Biometrika **36**, 142–148 (1949).

The author gives tables and diagrams for the distribution defined by the density $\psi(R)=2e^{-R}K_0(2e^{-\frac{1}{2}R})$, $R>0$. This distribution was previously [Ann. Math. Statistics **18**, 384–412 (1947); these Rev. **9**, 195] proved to be the asymptotic distribution of a certain linear transformation of the range of a sample from a symmetric distribution of so-called exponential type. *G. Elfving* (Ithaca, N. Y.).

Aspin, Alice A. Tables for use in comparisons whose accuracy involves two variances, separately estimated. Biometrika **36**, 290–293 (1949).

Let y be an estimate of a population parameter η, where the sampling variance of y is of the form $\lambda_1\sigma_1^2+\lambda_2\sigma_2^2$, where λ_1, λ_2 are known positive constants, and σ_1^2, σ_2^2 are two population variances. Let s_1^2, s_2^2 be estimates of σ_1^2, σ_2^2 based upon f_1 and f_2 degrees of freedom, respectively, and let them be distributed independently of each other and of y. Under the assumptions of normal theory, for a given probability level ϵ, critical values of $v=(y-\eta)/\sqrt{(\lambda_1s_1^2+\lambda_2s_2^2)}$ depend upon f_1, f_2 and also on the ratio $c=\lambda_1s_1^2/(\lambda_1s_1^2+\lambda_2s_2^2)$. The author offers two one-page tables (for $\epsilon=0.01$ and 0.05, respectively) for the four-argument function $v\{f_1, f_2, c, \epsilon\}$. The given values of f_1 and f_2 are 10, 12, 15, 20, 30, ∞, while those of c are 0.0 to 1.0 by tenths; v is given to two decimals (with values lying between 2.33 and 2.76, inclusive). The methods used in calculation are those of B. L. Welch [Biometrika **34**, 28–35 (1947); these Rev. **8**, 394] and the author [ibid. **35**, 88–96 (1948); these Rev. **9**, 600]. *A. A. Bennett.*

Welch, B. L. Further note on Mrs. Aspin's tables and on certain approximations to the tabled function. Biometrika **36**, 293–296 (1949).

This author discusses further the methods of calculation used in the paper reviewed above, and the reliability of the last given decimal for various parts of the tables. He discusses also the use of existing tables to estimate results for other probability levels. *A. A. Bennett.*

Bose, P. K. Incomplete probability integral tables connected with Studentised D^2-statistic. Calcutta Statist. Assoc. Bull. **2**, 131–137 (1949).

Tables are constructed to give the 5% right hand tail end points of the probability distribution of x for some values of ν, p, β, where x and β are transforms of the studentized D^2 statistic and Δ^2, respectively. Here $\Delta^2=\sum\alpha^{ij}(\alpha_i-\alpha_i')(\alpha_j-\alpha_j')$, where $\|\alpha_{ij}\|$ is the sample covariance matrix of two p variate normal populations with means α_i and α_i', respectively, and D is a corresponding statistic when samples of size n and n' are drawn from these populations; $\nu=n+n'-2$, $C^2=nn'p/(n+n')$, $x=C^2D^2/(n+n'+C^2D^2)$, $\beta=C^2\Delta^2/2$.

H. Chernoff (Urbana, Ill.).

Grubbs, Frank E. Sample criteria for testing outlying observations. Ann. Math. Statistics **21**, 27–58 (1950).

A detailed account is given of a criterion for testing whether, on the basis of a normal distribution, one or more extreme observations are consistent with the remaining observations of a sample. The criterion is the ratio of the sums of squares of deviations for the truncated and for the complete sample. Its distribution is obtained and tabulated for one extreme observation and for two extreme observations both at the same end; the criterion for one extreme at both ends is not examined. Other useful tables obtained during the course of the computations are presented. The efficiency and power of the suggested tests are not studied.

A. M. Mood (Santa Monica, Calif.).

Johnson, N. L. Systems of frequency curves generated by methods of translation. Biometrika **36**, 149–176 (1949).

The author is interested in the uses which may be made of transformations of variables such that the transformed variables may be considered to have a normal distribution. Let p_z be a probability function where $z=\gamma+\delta\log f(y)$ is distributed normally with unit variance, $f(y)\geqq 0$. If $f(y)=y$, the well known log-normal system, S_L, is obtained; if $f(y)=y/(1-y)$, $0<y<1$, a new system S_B; and if $f(y)=y+(y^2+1)^{\frac{1}{2}}$, $-\infty<y<\infty$, the new system S_U. The central moments and the measures of skewness and kurtosis are derived for S_B and S_U. The results are applied to the graduation of observed frequency distributions, and to the normalization of skewed distributions, particularly those of the Pearson system. *L. A. Aroian* (Culver City, Calif.).

Johnson, N. L. Bivariate distributions based on simple translation systems. Biometrika **36**, 297–304 (1949).

The author extends his investigation of univariate translation systems [see the preceding review] to bivariate distributions. Let $z=\gamma+\delta f(y)$, z a unit normal variate; f a function; γ, δ parameters; then if $f(y)=\log y$, the system is log-normal, s_N; if $f(y)=\log(y/1-y)$, s_B; and if $f(y)=\log\{y+(y^2+1)^{\frac{1}{2}}\}$, s_U. This gives rise to ten bivariate distributions whose characteristics are studied. A numerical example completes the study. *L. A. Aroian.*

Wishart, John. Cumulants of multivariate multinomial distributions. Biometrika 36, 47–58 (1949).

The author first derives for the ordinary Bernoulli multinomial distribution and the negative Bernoulli multinomial (the Pascal distribution) a cumulant recurrence relation due to S. Guldberg [Skand. Aktuartidskr. 18, 270–278 (1935)], preliminary to the derivation of corresponding cumulant recurrence relations for the multivariate Bernoulli and multivariate Pascal distribution. In the course of his work the cumulant generating functions, the cumulants including those of the fourth orders, and tables illustrating the method are given. In the multivariate case the formulas are too lengthy to be quoted here. *L. A. Aroian.*

Irwin, J. O. A note on the subdivision of χ^2 into components. Biometrika 36, 130–134 (1949).

The author shows by an orthogonal transformation that the χ^2 obtained when testing for independence within an $r \times s$ contingency table may be partitioned into the contributions from $(r-1)(s-1)$ fourfold tables.

D. G. Chapman (Seattle, Wash.).

Ghurye, S. G. Transformations of a binomial variate for the analysis of variance. J. Indian Soc. Agric. Statistics 2, 94–109 (1949).

The author extends slightly some theorems of Curtiss on the asymptotic properties of certain transformations of chance variables. He applies the extension to transformations of a binomial variate of which Bartlett's z is a special case. The variance of the transformed variate is computed for several cases where $n=10$, $n=15$, $n=20$. A transformation which gives rather constant variance as a function of p for $n=15$ is given by

$$T(x) = n^{\frac{1}{2}} \sin^{-1}(\alpha/n)^{\frac{1}{2}}, \quad n^{\frac{1}{2}} \sin^{-1}(1-\alpha/n)^{\frac{1}{2}}, \quad n^{\frac{1}{2}} \sin^{-1} X^{\frac{1}{2}},$$

for $X=0$, $X=1$, and $0<X<1$, respectively, where X is the observed proportion of successes and $\alpha=.2$.

H. Chernoff (Urbana, Ill.).

Thomas, Marjorie. A generalization of Poisson's binomial limit for use in ecology. Biometrika 36, 18–25 (1949).

Let x be a Poisson distributed random variable with mean m and y a random variable such that the conditional distribution of $y-x$ given x is Poisson with mean λx. Reasons, theoretical and empirical, are given for supposing that the number of plants of a given species in a square chosen at random from a homogeneous ecological area will be distributed approximately like y for some value of the parameters m and λ. Elementary mathematical facts about this family of distributions are deduced and two practical methods for estimating the parameters are explored.

L. J. Savage (Chicago, Ill.).

Aitken, A. C. On the Wishart distribution in statistics. Biometrika 36, 59–62 (1949).

The author discusses various proofs and some modifications of Wishart's distribution and shows that all of them essentially involve a lemma of Siegel, which he proves. The lemma is as follows: given S, an arbitrary positive definite real symmetric matrix, and T, a variable positive definite real symmetric matrix, both of order $k \times k$; integration being over the domain of positive-definiteness of T, it is asserted that

$$\int \exp(-\operatorname{tr} ST)\,|T|^{n-\frac{1}{2}(k+1)}\,dt = \{\Gamma(\tfrac{1}{2})\}^{\frac{1}{2}k(k+1)} \prod_{h=0}^{k-1} \Gamma(n-\tfrac{1}{2}h)\,|S|^{-n}.$$

L. A. Aroian (Culver City, Calif.).

Lehmann, E. L. Some principles of the theory of testing hypotheses. Ann. Math. Statistics 21, 1–26 (1950).

For a class Ω of distributions θ over a space X, a test of the hypothesis $H: \{\theta \varepsilon \omega \subset \Omega\}$ is a function $\phi(x)$, $0 \leqq \phi(x) \leqq 1$, specifying the probability with which H is rejected when x is observed. The power of ϕ is the function $E_\theta(\phi)$, the probability of rejection as a function of θ. Various reasonable principles for selecting a test are considered, some of their properties are developed, and some illustrations are given, in which use of these principles reduces the set of available tests to a narrow class. The principles considered are as follows. (1) Likelihood ratio. The Neyman-Pearson theorem shows that this test solves the problem completely for testing a simple hypothesis against a single alternative. An example is given, testing a simple hypothesis against a class of alternatives, in which the likelihood ratio test is worse than useless. Under certain conditions, essentially the case of one-sided alternative values of a real parameter, there is a mixture λ of the alternative distributions, the "least favorable" distributions of Wald, such that the likelihood ratio test of the simple hypothesis against λ is best against each alternative. (2) Sufficiency. For any test $\phi(x)$ and any sufficient statistic T, the test $E[\phi \mid T]$ is equivalent to ϕ, i.e., has the same power for all θ. Thus only tests depending on sufficient statistics need be considered. In certain cases, minimal sufficient statistics exist. (3) Invariance. For any group of transformations leaving the statistical problem invariant, such as a change of scale or permutation of independent identical observations, ϕ should be invariant. This principle is closely related to that of sufficiency. (4) Unbiasedness.

$$\max_{\theta \varepsilon \omega} E_\theta \phi \leqq \min_{\theta \varepsilon \Omega - \omega} E_\theta \phi.$$

An extension of unbiasedness is that $E_\theta \phi$ should increase with the "distance" of θ from ω. (5) Stringency. In the class of ϕ with $E_\theta \phi \leqq \alpha$ for $\theta \varepsilon \omega$, a ϕ is called most stringent if it minimizes $\sup[h(\theta) - E_\theta \phi]$ over $\theta \varepsilon \Omega - \omega$, where $h(\theta) = \sup E_\theta \phi$ over all ϕ in the given class. A theorem of Hunt and Stein asserts that, for certain groups G of transformations, if the problem is invariant under G, there is a most stringent test invariant under G. (6) Minimizing maximum loss. If $w(\theta)$ is the loss in accepting H when the true distribution is θ, minimize $\max w(\theta)[1 - E_\theta \phi]$ for $\theta \varepsilon \omega$, subject to $\max_{\theta \varepsilon \omega} E_\theta \phi \leqq \alpha$. For a suitable loss function, this notion reduces to that of most stringent, and the theorem of Hunt and Stein can be extended to this case.

D. Blackwell (Washington, D. C.).

Kvit, I. D. On N. V. Smirnov's theorem concerning the comparison of two samples. Doklady Akad. Nauk SSSR (N.S.) 71, 229–231 (1950). (Russian)

Let $S_m(x)$ and $T_n(x)$ be the empirical distribution functions of two samples of mutually independent random variables having a common continuous distribution function $F(x)$, the samples being of size m and n, respectively. Let $0<\theta_1<\theta_2<1$ and define α_{mn} and β_{mn} by $\alpha_{mn} = \min_s [S_m(x) = \theta_1;\ T_n(x) = \theta_1]$, $\beta_{mn} = \max_s [S_m(x) = \theta_2;\ T_n(x) = \theta_2]$. The author gives the limiting distributions of

$$D^{+}_{mn}(\theta_1, \theta_2) = \sup_{\alpha_{mn} \leqq x \leqq \beta_{mn}} \{S_m(x) - T_n(x)\}$$

and

$$D_{mn}(\theta_1, \theta_2) = \sup_{\alpha_{mn} \leqq x \leqq \beta_{mn}} |S_m(x) - T_n(x)|$$

as $N = [mn/(m+n)] \to \infty$, $m/n =$ constant, thus generalizing Smirnov's theorem [Bull. Math. Univ. Moscou 2, no. 2

(1939); these Rev. 1, 345; see also Feller, Ann. Math. Statistics 19, 177–189 (1948); these Rev. 9, 599] dealing with the case $\theta_1=0$, $\theta_2=1$. The limiting distributions (using $N^{\frac{1}{2}}$ as a normalizing factor) are the same as those given by Maniya [Doklady Akad. Nauk SSSR (N.S.) 69, 495–497 (1949); these Rev. 11, 261] for the one-sample case, $T_n(x)=F(x)$. *G. E. Noether* (New York, N. Y.).

Barankin, E. W. Locally best unbiased estimates. Ann. Math. Statistics 20, 477–501 (1949).

Let P be a class of probability measures u_θ on a space X which have density $\Pi_\theta(x)$ with respect to a particular measure u_{θ_0}. A function $\phi(x)$ is an unbiased estimate of $g(\theta)$ if $E_\theta\phi=g$ for all θ. Without loss of generality, suppose $g(\theta_0)=0$. The author considers the problem of finding the unbiased estimate for which $E_{\theta_0}|\phi|^s$ has its minimum value λ_0, $1<s\leqq\infty$. For $s<\infty$, a necessary and sufficient condition that there is an unbiased estimate ϕ of g with $E_{\theta_0}|\phi|<\infty$ is that there is a constant C such that $|\sum a_i g(\theta_i)|\leqq C\|\sum a_i\Pi_{\theta_i}\|_r$, where the norm $\|\ \|_r$ is that for L_r over X with respect to u_{θ_0}, and $1/r+1/s=1$. If there is such an unbiased estimate, there is exactly one, say ϕ_0, with $\lambda=\lambda_0$; λ_0 is the lower bound of admissible C. The necessity follows from Hölder's inequality; the sufficiency is an application of the Hahn-Banach theorem and the representation theorem for linear functionals on L_r. For $s=\infty$, the only change is that ϕ_0 need not be unique. That λ_0 is the lower bound of admissible C's makes it possible to evaluate λ_0 in certain cases. In particular, the lower bounds for the variance of unbiased estimates given by Cramér and Rao and by Bhattacharyya may be obtained in this way. If there is an unbiased estimate with finite λ, a sequence of functions ϕ_n is defined such that $\|\phi_n-\phi_0\|\to 0$. A special result for $s=2$ is that, if ϕ_0 exists, it is the only unbiased estimate in the closed linear manifold determined by the functions $\Pi_\theta(x)$.

D. Blackwell (Washington, D. C.).

Hemelrijk, J. Construction of a confidence region for a line. Nederl. Akad. Wetensch., Proc. 52, 995–1005 = Indagationes Math. 11, 374–384 (1949).

Let $\{P_i\}$ $(i=1,\cdots,n)$ be a set of random points in a plane V such that $P_i=Q_i+w_i$, where $Q_1,\cdots,Q_n$ are fixed points of V situated on a line L and $w_1,\cdots,w_n$ are independently distributed random vectors (called errors). The points $Q_1,\cdots,Q_n$ and the line L are unknown. Under weak restrictions on the distribution of the errors, the author derives confidence sets for the unknown line L, for the slope of L, and for the intercept of L on the ordinate axis. These confidence sets depend only on $P_1,\cdots,P_n$. The solution obtained and methods used by the author are different from those given by the reviewer in treating essentially the same problem [Ann. Math. Statistics 11, 285–300 (1940); these Rev. 2, 108]. The author's results remain valid under rather general conditions and do not depend on the particular shape of the distribution of the errors.

A. Wald (New York, N. Y.).

Wald, A., and Wolfowitz, J. Bayes solutions of sequential decision problems. Ann. Math. Statistics 21, 82–99 (1950).

This paper derives and extends theorems on the existence and characterization of Bayes solutions announced by the authors in a paper with the same title [Proc. Nat. Acad. Sci. U. S. A. 35, 99–102 (1949); these Rev. 10, 466]. It overlaps a paper by Arrow, Blackwell, and Girshick [Econometrica 17, 213–244 (1949); these Rev. 11, 261]. Remarks in both papers clarify the allocation of priority. The paper under review goes further in the study of infinite sets of distributions than that of Arrow, Blackwell, and Girshick, but on the whole it tends to be less abstract and general. A technical lacuna regarding measurability overlooked by the authors [same Ann. 19, 326–339 (1948); these Rev. 10, 201] is filled in here. Cf. the review of Arrow, Blackwell, and Girshick cited above. *L. J. Savage* (Chicago, Ill.).

Anscombe, F. J. Large-sample theory of sequential estimation. Biometrika 36, 455–458 (1949).

Let $z_1, z_2, \cdots$ be independent identical variables with unknown mean θ and variance $v(\theta)$. The author considers the sequential estimation procedure defined by a function $k(m)$; sampling stops as soon as $Z_m=z_1+\cdots+z_m\geqq k(m)$, and θ is estimated by $\hat{\theta}=k(n)/n$, where n is the sample size obtained. For a sequence of tests $k(m)$ satisfying certain regularity conditions and with $E(n)$ becoming infinite, $\hat{\theta}$ is asymptotically normal, with mean θ and variance $v(\theta)/n_0$, where n_0 is the (unique by regularity conditions) sample size such that $\theta n_0=k(n_0)$. *D. Blackwell* (Washington, D. C.).

Finney, D. J. On a method of estimating frequencies. Biometrika 36, 233–234 (1949).

Haldane [Biometrika 33, 222–225 (1945); these Rev. 8, 477] obtained an unbiased estimate of the binomial parameter p when the sampling is "inverse," i.e., sampling is continued until a specified number of "defectives" has been obtained. In this paper an unbiased estimate of the variance of Haldane's estimate is obtained, and its use in designing the inverse sampling procedure discussed. A simple method is indicated for determining confidence limits for p from the same tables that are used with direct sampling.

D. Chapman (Seattle, Wash.).

Hartley, H. O. Tests of significance in harmonic analysis. Biometrika 36, 194–201 (1949).

Let $\{y_i\}$ be n observations taken at times $2\pi j/n$, $j=0,\cdots,n-1$. The author fits a curve

$$Y_i=\sum_{i=1}^{m}[a_i\cos itc+b_i\sin itc]+a_0,\qquad c=2\pi/n,$$

with the usual formulas for a_i and b_i, and wishes to test for the significance of $\max_i(a_i^2+b_i^2)$. A Studentized test, based on the assumption that the y_i's are mutually independent, and normally distributed with mean 0 and common unknown variance, is used. The usefulness of this test is investigated under various alternative assumptions on the character of the y_i distributions. *J. L. Doob* (Ithaca, N. Y.).

Huzurbazar, V. S. On a property of distributions admitting sufficient statistics. Biometrika 36, 71–74 (1949).

If $L(x,\theta_j)$ is the likelihood function of a random Cartesian point x for the parameters θ_j admitting as few regular sufficient statistics as parameters, then, it is shown, the expectation of a partial derivative of any order is equal to the value of the corresponding derivative at that value $x(\theta_j)$ which satisfies the likelihood equations. From this result applied to second derivatives it is further shown under mild restrictions that the likelihood equations have a unique solution in θ_j for each x and that this solution does actually maximize the likelihood. *L. J. Savage* (Chicago, Ill.).

Nandi, H. K. A note on conditional tests of significance. Bull. Calcutta Math. Soc. **41**, 121–124 (1949).

Denote by x, y, θ, θ' the respective vectors $(x_1, \cdots, x_p)$, $(y_1, \cdots, y_n)$, $(\theta_1, \cdots, \theta_k)$, $(\theta_{k+1}, \cdots, \theta_{k+l})$ and assume that $\phi(x, y, \theta, \theta')$, the joint probability density function of x and y, can be written as $\phi_1(x, \theta)\phi_2(y|x, \theta')$, where $\phi_2(y|x, \theta')$ is the conditional distribution of y for fixed x. The author studies conditions under which a test of a hypothesis regarding θ' should be independent of the distribution of x. When

$$\phi_2(y|x) = \sigma^{-1}(2\pi)^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}\sigma^{-2}\{y - a - \sum_{i=1}^{p}\beta_i x_i\}^2\right],$$

a sufficient condition is given for the power function of the standard test of the hypothesis $\beta_i = 0$ $(i = 1, \cdots, p)$ to be independent of the distribution of x. *E. Paulson.*

Anderson, R. L., and Anderson, T. W. Distribution of the circular serial correlation coefficient for residuals from a fitted Fourier series. Ann. Math. Statistics **21**, 59–81 (1950).

Given a time-series x_t which has a seasonal variation with normal disturbance z_t, the independence of the z_t against the alternative hypotheses of a linear regression of the type $z_t = \rho z_{t-L} + u_t$ $(\rho \neq 0)$ is tested by the likelihood ratio criterion, which is shown to be equivalent to the circular coefficient, R_L. Exact and approximate distributions of R_L are deduced, and for $L = 1$ its significance points are tabulated for the cases of semi-annual, quarterly, bi-monthly and monthly data. *H. Wold* (Uppsala).

Teatini, Ugo. Interpolazione di serie cicliche. Statistica, Milano **8**, 125–140 (1948).

The author discusses by elementary methods the least squares fit of a circle to cyclical time series.

E. Lukacs (Washington, D. C.).

Marseguerra, Elio. Su alcuni valori caratteristici delle serie cicliche trattate con metodo continuo. Metron **15**, 71–115 (1949).

The author studies continuous cyclical time series by purely descriptive methods. He defines and discusses several indices useful in his work. *E. Lukacs.*

Mathematical Economics

Hickman, W. Braddock. The determinacy of absolute prices in classical economic theory. Econometrica **18**, 9–20 (1950).

This is one of three articles in criticism of Patinkin's argument that classical economic systems are inconsistent [Econometrica **17**, 1–27 (1949); these Rev. **10**, 725]. The classical system may be expressed by:

$$D_i = f_i(p_1, \cdots, p_{n-1}), \tag{1}$$

$$S_i = g_i(p_1, \cdots, p_{n-1}), \tag{2}$$

$$X_i = D_i(p) - S_i(p) = 0 \tag{3}$$

$(i = 1, \cdots, n)$, where the first $n-1$ equations refer to commodities, and the nth to money. The equations (3) are the equilibrium conditions. The first assumption of such a system is that the supply and demand functions for commodities are homogeneous of degree zero. The monetary equations may be written:

$$(1)'\ D_n = f_n(p), \quad (2)'\ S_n = g_n(p), \quad (3)'\ X_n = f_n(p) - g_n(p),$$

where p is a price vector. It follows that (1)′–(3)′ are homogeneous of degree one, and the equilibrium conditions (3) are dependent. Hence either the $n-1$ commodity equations in (3) are inconsistent and no equilibrium solution exists or these equations are consistent and (3)′ is redundant as an equilibrium condition. The second classical assumption is that the commodity equations are consistent. Since any set of $n-1$ absolute prices reducible to the $n-2$ equilibrium price ratios is a solution, the price level is indeterminate in the real sector. Hence, the third assumption made is that a constraint, e.g., the Cambridge equation, in the monetary sector determines the absolute price level. The author gives an example of a consistent classical system.

Patinkin's argument that no consistent system can be constructed is based on two assertions: (1) the assumption of a monetary constraint is incompatible with the homogeneity assumption and (2) the homogeneity assumption contradicts the assumption of consistency. Both of these propositions are shown to be false, the basic flaw in the argument being the attempt to establish consistency by counting equations and unknowns. The article closes with a discussion of dependence of equations in economic systems.

M. P. Stoltz (Providence, R. I.).

Leontief, Wassily. The consistency of the classical theory of money and prices. Econometrica **18**, 21–24 (1950).

The author states that Patinkin's argument [see the preceding review] is in error because he overlooked the interdependence of the commodity functions. Any one of the $2n-2$ supply and demand functions for commodities [equations (2) and (3) in the preceding review] can be derived from the remaining $2n-3$ functions. In classical systems this dependence is ordinarily guaranteed by the insertion of additional constraints, e.g., Say's law. A simple illustration of a consistent system involving linear dependence is given.

M. P. Stoltz (Providence, R. I.).

Phipps, Cecil G. A note on Patinkin's "Relative prices." Econometrica **18**, 25–26 (1950).

Several deductions are drawn from the joint assumptions of perfect competition and utility maximization as a behavior rule. They are summarized by the statement: if the relative price of the good is zero, it has no marginal utility. It is now argued that Patinkin [see the two preceding reviews] finds the classical system inconsistent because of contradictory assumptions. It is assumed first that the stock of money does not enter into the utility function since money does not have direct utility. It is then assumed that the price of money relative to goods is unity, which contradicts the first assumption. The author's criticism seems invalid to the reviewer. *M. P. Stoltz* (Providence, R. I.).

Arrow, Kenneth J. Homogeneous systems in mathematical economics: A comment. Econometrica **18**, 60–62 (1950).

The author presents alternative proofs of two theorems on homogeneity due to Tintner [Econometrica **16**, 273–294 (1948); these Rev. **10**, 556] which were derived under the assumption that the functions involved were differentiable twice. It is now shown that these results can be obtained without any explicit assumption of differentiability. It may be noted that, in the proofs, the conditions for a relative constrained maximum are assumed to be satisfied.

M. P. Stoltz (Providence, R. I.).

Amoroso, Luigi. Revisione della teoria matematica dell' interesse. Atti Ist. Naz. Assicuraz. **14**, 83–102 (1942).

Denote by $y(t)$, $t_0 \leqq t \leqq t_1$, an individual's income per time unit, at time t. Write $\eta(t) = y(t)/P(t)$, where $P(t)$ is the price level. Let his investment opportunities, aside from borrowing and lending, be expressed by the integral equation in $\eta(t)$,

$$\text{(A)} \qquad \int_{t_0}^{t_1} F[\eta(t), \eta'(t)]dt = 0,$$

where $\eta'(t) = d\eta/dt$. Denote by $x(s, t)$ his money receipts (payments if negative) per time unit at time t, in consequence of loans contracted at time s. Write $\xi(s, t) = x(s, t)/P(t)$. Then, for every s, $t_0 \leqq s \leqq t \leqq t_1$,

$$\text{(B)} \qquad \int_s^{t_1} P(t)\xi(s, t)e^{-\int_s^t i(u)du}dt = 0,$$

where $i(u)$ is the interest rate at time u. The individual chooses the functions η, ξ so as to maximize, subject to (A) and (B) and to certain boundary conditions, the expression

$$\text{(C)} \qquad \int_{t_0}^{t_1} dt \int_{t_0}^{t} \Phi(\xi+\eta, \xi_1, \xi_1+\eta')ds,$$

say, where $\xi_1 = \partial\xi/\partial s$, $\xi_2 = \partial\xi/\partial t$. Denote the optimal functions for the αth individual by $\xi^{(\alpha)}$, $\eta^{(\alpha)}$. They are seen to depend upon the interest rate $i(t)$; this function of time is common to all individuals and is determined by the condition that, per unit of time, the excess of all receipts over all payments equal the amount of money $M(t)$ created (or destroyed) by the banks:

$$\text{(D)} \qquad \sum_\alpha \int_{t_0}^{t} \xi^{(\alpha)}(t, s)P(s)ds = M(t).$$

The model is reduced to the more special one of I. Fisher [The Theory of Interest . . . , Macmillan, New York, 1930], (1) by disregarding the distinction, made in (B), between long- and short-term loans; (2) by assuming that the opportunity and utility functions corresponding, respectively, to F in (A) and Φ in (B), are independent of the derivatives of η and ξ; (3) by regarding time as a discrete variable; (4) by assuming the price level P constant over time; (5) by regarding banks as ordinary utility-maximizing lenders rather than as makers of money, led by purposes of price policy. The general model is solved as a problem in the calculus of variations. *J. Marschak* (Chicago, Ill.).

TOPOLOGY

Whitney, Hassler. Relations between the second and third homotopy groups of a simply connected space. Ann. of Math. (2) **50**, 180–202 (1949).

Es werden spezielle Abbildungen der Sphäre S^3 betrachtet: in S^3 sind endlich viele disjunkte Volltorusse ("tubes") T_i ausgezeichnet, die in bestimmter Weise als topologische Bilder des Produktes $S^1 \times \sigma^2$ gegeben sind, wobei S^1 eine Kreislinie, σ^2 eine Kreisscheibe bezeichnen; f sei eine solche Abbildung von S^3 in den einfach zusammenhängenden Raum R, dass $f(S^3 - \sum T_i)$ ein Punkt ist; sie bestimmt ein Element $\xi \in \pi_3(R)$; ferner bestimmt die durch f bewirkte Abbildung einer Schnittfläche in T_i, welche einer Kreisscheibe $p \times \sigma^2$ entspricht (p Punkt auf S^1), ein Element $\alpha_i \in \pi_2(R)$. Es entsteht die Aufgabe, ξ durch die α_i auszudrücken. Dies geschieht mit Hilfe der Whitehead'schen Produkte $\alpha_i\alpha_j$ und der mit diesen zusammenhängenden Operation $\eta(\alpha)$, die jedem $\alpha \in \pi_2(R)$ dasjenige $\eta(\alpha) \in \pi_3(R)$ zuordnet, das entsteht, wenn man erst die Standard-Faser-Abbildung $S^3 \to S^2$ und dann diejenige Abbildung $S^2 \to R$ ausübt, welche α bestimmt [cf. J. H. C. Whitehead, Ann. of Math. (2) **42**, 409–428 (1941); diese Rev. **2**, 323]; die Produkte und die η-Operation werden übrigens ohne Benutzung der Whitehead'schen Arbeit im Rahmen der Betrachtung der Torus-Abbildungen neu eingeführt.

Jeder Torus T_i ist in die Bilder der Kreislinien $S^1 \times q$ (q Punkt auf σ^2) gefasert; die Verschlingungszahl zweier Fasern in T_i heisse v_{ii}, einer Faser von T_i mit einer Faser von T_j: v_{ij} ($i \neq j$). Dann lautet die gesuchte Formel $\xi = \sum_i v_{ii}\eta(\alpha_i) + \sum_{i<j} v_{ij}\alpha_i\alpha_j$; da $\alpha\alpha = 2\eta(\alpha)$ ist, folgt hieraus auch $2\xi = \sum v_{ij}\alpha_i\alpha_j$, wobei über alle i, j zu summieren ist; hier darf man aber nicht durch 2 dividieren, falls $\pi_3(R)$ Elemente von endlicher gerader Ordnung enthält. Der Formel wird noch eine andere Gestalt gegeben, wobei ein Graph herangezogen wird, der sich durch Projektion des Torus-Systems in eine Ebene ergibt. Für den Fall, dass $\pi_3(R)$ Elemente endlicher Ordnung enthält, wird noch eine weitere Konstruktion vorgenommen, die darin besteht, dass in die S^3 gewisse Zylinder, "junctions" genannt, eingebaut werden; die Einzelheiten dieser Konstruktion und die sich dann ergebende Formel für ξ sind zu kompliziert, als dass ich sie hier beschreiben könnte. Die Ergebnisse dieser Arbeit sollen als Hilfsmittel in zwei anderen Arbeiten des Verfassers Verwendung finden [man vergl. die zwei nachstehenden Referate]. *H. Hopf* (Zürich).

Whitney, Hassler. Classification of the mappings of a 3-complex into a simply connected space. Ann. of Math. (2) **50**, 270–284 (1949).

Soit K un complexe à 3 dimensions, et R un espace connexe (par arcs) et simplement connexe, tel que $\pi_2(R) \approx H_2(R)$ soit engendré par un nombre fini d'éléments. Il s'agit de caractériser par des invariants homologiques les classes d'applications (continues) de K dans R; on sait que deux applications sont dites dans la même "classe" si elles sont homotopes.

Il est connu qu'une application f de K dans R définit un élément W_f du groupe de cohomologie $H^3(K, \pi_2(R))$, qui ne dépend que de la classe de f. D'où une application canonique φ de l'ensemble des classes d'applications, sur le groupe $H^3(K, \pi_2)$. Étant donné un élément W de $H^3(K, \pi_2)$, il s'agit de déterminer l'image réciproque $\varphi^{-1}(W)$. L'auteur montre que l'ensemble $\varphi^{-1}(W)$ est en correspondance biunivoque avec le quotient du groupe $H^2(K, \pi_2(R))$ par un sous-groupe G_W défini comme suit. Grâce à l'application bilinéaire (connue) de $\pi_2 \times \pi_2$ dans π_3, et en se servant d'un mémoire antérieur [voir l'analyse ci-dessus], il semble que l'auteur définisse un homomorphisme α de $H^1(K, \pi_2(R))$ dans $H^3(K, \pi_2(R))$ (mais il ne dit pas si α est indépendant de la décomposition, qu'il utilise, de π_2 comme somme directe de groupes cycliques). Ici α est nul quand π_2 n'a pas d'élément d'ordre pair. D'autre part, le cup product définit une application bilinéaire $H^1(K, \pi_2) \times H^2(K, \pi_2) \to H^3(K, \pi_3)$; donc chaque $W \in H^2(K, \pi_2)$ définit un homomorphisme β_W de $H^1(K, \pi_2)$ dans $H^3(K, \pi_3)$. Le sous-groupe G_W dont il a été question ci-dessus est l'image de $H^1(K, \pi_2)$ par l'homomorphisme $\alpha - \beta_W$. *H. Cartan* (Paris).

Whitney, Hassler. An extension theorem for mappings into simply connected spaces. Ann. of Math. (2) **50**, 285–296 (1949).

Es sei: R ein einfach zusammenhängender Raum, dessen Gruppe π_2 eine endliche Basis besitzt; $K = K^4$ ein Komplex,

K^i sei i-dimensionales Gerüst; f eine Abbildung von K^2 in R. Da R einfach zusammenhängend ist, dürfen wir annehmen, dass $f(K^1)$ ein Punkt ist; auf K^2 dürfen wir, nach Vornahme einer stetigen Abänderung, f so wählen, dass sie eine auf bestimmte Weise definierte "Standard"-Abbildung ist, die von der Auszeichnung einer Basis in $\pi_2(R)$ abhängt. Jedem Simplex von K^2 entspricht ein Element von $\pi_2(R)$, und es ist somit eine Kette W_f^2 mit Koeffizienten aus $\pi_2(R)$ erklärt; da f auch auf K^3 definiert ist, ist W_f^3 ein Cozyklus. Das Bild des Randes jedes 4-dimensionalen Simplexes bestimmt ein Element von $\pi_3(R)$, und damit ist ein Cozyklus W_f^4 mit Koeffizienten aus $\pi_3(R)$ erklärt, das "2. Hindernis", das sich der Erweiterung von f entgegensetzt. Es handelt sich darum, W_f^4 durch W_f^3 auszudrücken. Dies geschieht durch eine Formel, die besonders einfach ist, falls π_3 kein Element endlicher gerader Ordnung enthält; in ihr treten die Whitehead'schen Produkte von Elementen der Gruppe π_2 auf, sowie in dem Fall, dass π_2 Elemente gerader Ordnung enthält, die Steenrod'schen $\cup_i$-Produkte mit $i=1, 2$. Die Beweise der Formeln geschehen durch rein geometrische Überlegungen und sind kompliziert, besonders im Falle der Existenz von Elementen gerader Ordnung in π_2. Die beiden vorhergehenden Arbeiten des Verfassers werden benutzt [man vergl. die zwei vorstehenden Referate]. In der Einleitung weist der Verfasser auf den Zusammenhang seiner Fragestellung und seines Ergebnisses mit zwei älteren Arbeiten von Pontrjagin hin. *H. Hopf* (Zürich).

*Whitney, Hassler. **La topologie algébrique et la théorie de l'intégration.** Topologie algébrique, pp. 107–113. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12. Centre de la Recherche Scientifique, Paris, 1949. 600 francs.

The main results have been announced in an earlier note [Proc. Nat. Acad. Sci. U. S. A. **33**, 1–6 (1947); these Rev. **8**, 397]. *S. Chern* (Chicago, Ill.).

Pontryagin, L. S. Vector fields on manifolds. Amer. Math. Soc. Translation no. **13**, 60 pp. (1949).

Translated from Mat. Sbornik N.S. **24(66)**, 129–162 (1949); these Rev. **10**, 727.

Denjoy, Arnaud. Les domaines d'approximation régulière dans les espaces cartésiens. C. R. Acad. Sci. Paris **230**, 885–888 (1950).

The author considers a regular polyhedral division of Euclidean n-space and makes a study of the boundary ϕ of a set of segments of this division. Cyclic fragments of such boundaries ϕ are defined and a result on the distribution of multiple elements of ϕ obtained. From this it follows that no such multiple elements can exist in the case of a domain of approximation for a plane continuum so that the boundary of such a domain is a simple polygon.

G. T. Whyburn (Charlottesville, Va.).

*de Rham, Georges. **Sur les conditions d'homéomorphie de deux rotations de la sphère à n dimensions, et sur les complexes avec automorphismes.** Topologie algébrique, pp. 87–95. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12. Centre de la Recherche Scientifique, Paris, 1949. 600 francs.

Two maps R and R' of a manifold V into itself are said to be homeomorphic if there exists a topological map T of V onto itself such that $TR=R'T$. Three questions are raised. (1) What is the condition for a topological map R of S^n onto itself to be homeomorphic with a rotation? A necessary condition is that the collection $\{R^p\}$ for all integers p be equi-continuous. (2) What is the condition for two rotations to be homeomorphic? A sufficient condition is that they have the same characteristic roots. (3) Given two rotations R, R', consider the maps f such that $fR=R'f$. These maps are divided into homotopy classes relative to R and R'. It is demanded to enumerate and to characterize these classes. The last problem was studied by Rueff, Franz, and Hirsch in the particular case where R and R' are of finite order h and all of their characteristic roots are primitive hth roots of unity. In the present paper, a partly different approach to their results is given in the same particular case of an odd sphere. For the second problem, it is shown that two rotations (satisfying the above conditions) have the same characteristic roots if they are homeomorphic in the combinatory sense. *S. T. Hu.*

Murty, A. S. N. Simply ordered spaces. J. Indian Math. Soc. (N.S.) **13**, 152–158 (1949).

The author proves the following properties of a simply ordered space R stated in the literature [see Birkhoff, Lattice Theory, 1st ed., Amer. Math. Soc. Colloquium Publ., v. 25, 1940, p. 28; these Rev. 1, 325; Alexandroff and Hopf, Topologie, Springer, Berlin, 1935, p. 92, Aufgabe]: R is normal and is bicompact when it has two end elements and every section is a cut. (In the latter case he proves only that R is compact, not necessarily bicompact.) The property that every section of R is a cut is also proved to be equivalent to the connectedness of R. *H. Tong.*

GEOMETRY

*Zimmermann, Franz. **Ein System von Ordnungsaxiomen für den euklidschen Raum und seine n-dimensionale Verallgemeinerung.** Thesis, Eidgenössische Technische Hochschule in Zürich, 1949. 79 pp.

This thesis offers in a detailed and thorough manner a new foundation for n-dimensional Euclidean geometry in terms of a single class of undefined elements, "points," and one primitive relation, "equal orientation." All incidence relations, for example, are defined in terms of this primitive one, and incidence properties, which feature as postulates in other systems [e.g., Hilbert's Grundlagen] are proved here as theorems. If $a, b, \cdots$ denote points and ab/cd symbolizes the relation "the pair a, b has the same orientation as the pair c, d," one-dimensional geometry is characterized by the axioms (1) ab/cd implies $c\neq d$, (2) ab/bc implies ab/ac, (3) ab/xy & cd/xy implies ab/cd, and (4) if $a\neq b$ & $c\neq d$ then ab/cd or ba/cd. If p, a, b are points and if a point x exists such that px/ab or xp/ab, then p is said to be on the "line" (a, b). Orientation in the plane is established by postulates on triples of points (e.g., if abc/def then d does not lie on (e, f); abc/def implies bca/def). The closure axiom for plane geometry is the assumption that if there exists a point s which is on neither of the lines (x, y), (a, b), then xab/yab. The system for E_3 consists of 18 postulates, the last six of which orient three-space by assumptions concerning equally oriented quadruples. Independence proofs are given.

L. M. Blumenthal (Columbia, Mo.).

Menger, Karl. The projective space. Duke Math. J. 17, 1–14 (1950).

Bereits in früheren Arbeiten hat sich Verf. mit einer axiomatischen Begründung der projektiven Geometrie befasst, die das Dualitätsprinzip in Ebene und Raum in Evidenz setzt [siehe z.B. Rice Inst. Pamphlet 27, 41–79 (1940); C. R. Acad. Sci. Paris 228, 1273–1274 (1949); Rep. Math. Colloquium (2) 8, 81–87 (1948); diese Rev. 3, 181; 10, 618]. Hier gibt Verf. für die Ebene ein System von 5, für den Raum ein System von 10 Axiomen, darunter je 2 Existenzaxiome. In diesen sind nicht inzidente, zueinander duale Raumelemente zu Paarungen (sog. opposition) zusammengefasst, in der Ebene die Paare Punkt-Gerade, im Raum die Paare Punkt-Ebene bzw. Gerade-Gerade. Zwei solche Paare (P, g) und (Q, k) in der Ebene heissen unverkettet (unbalanced), wenn die Verbindungsgerade von P, Q nicht durch den Schnittpunkt von g, k geht, andernfalls verkettet (balanced). Die Existenz von zwei verketteten und zwei unverketteten Paarungen wird axiomatisch gefordert. Analog heisst im Raum ein Tripel von nichtkollinearen Paarungen (P, α), (Q, β), (R, γ) unverkettet, wenn die Verbindungsebene von P, Q, R nicht durch den Schnittpunkt von α, β, γ hindurchgeht, andernfalls verkettet. Die Existenz eines unverketteten und eines verketteten Tripels wird axiomatisch gefordert. Der Begriff des vollständigen n-Ecks und n-Seits wird in den dualen Begriff des n-Zyklus zusammengefasst. Der Satz von Desargues in der Ebene lässt sich dann in selbst dualer Form als adäquate aussagen über konjugierte 4-er-Zyklen aussprechen. Er gilt in jedem ebenen System des räumlichen Systems. Die Elemente eines Ebenenbündels resp. Ebenenbüschels und die dazu dualen Gebilde gestatten eine umkehrbar eindeutige Zuordnung. Sämtliche Axiome sind erfüllt in der analytischen Geometrie homogener Koordinaten über einem Schiefkörper. Der räumliche Pascal'sche Satz (Dandelins Hexagramme mystique) lässt sich in selbstdualer Form nach Einführung des Begriffes Geradenzyklus aussprechen. In dem vorliegenden Axiomensystem der Ebene und des Raumes ist jedes Axiom von den übrigen unabhängig. *R. Moufang.*

Blumenthal, L. M., and Kelly, L. M. New metric-theoretic properties of elliptic space. Univ. Nac. Tucumán. Revista A. 7, 81–107 (1949).

This paper continues the metric study of elliptic space started by the first named author in two recent papers [Trans. Amer. Math. Soc. 59, 381–400 (1946); 62, 431–451 (1947); these Rev. 8, 82; 9, 457]. It gives a characterization of pseudo-$E_{1,r}$ quadruples imbeddable in $E_{2,r}$. A "cross" in $E_{n,r}$ is defined as the locus of points equidistant from two given points. An "orthocentric quadruple" is a system of four points p_1, p_2, p_3, p_4 such that $E_{1,r}(p_i, p_j)$ and $E_{1,r}(p_k, p_l)$ are mutually perpendicular where i, j, k, l is any permutation of 1, 2, 3, 4. A "metric basis" in a semimetric space is a set of points for which there exists no two points of the space with the same (ordered) distances to the points of the set. Typical theorems are as follows. An orthocentric quadruple is freely movable. Five points of a cross of the $E_{2,r}$, no four of which are orthocentric, are freely movable, ("Crowding theorem.") Every set of eight points of the $E_{2,r}$ contains a triple with perimeter less than πr, and eight is the smallest number with this property. From an earlier proposition a triple of $E_{2,r}$ with perimeter less than $\frac{4}{3}\pi r$ is freely movable. Using the "crowding theorem" as an essential tool the authors prove that a congruence order of $E_{2,r}$ with respect to the class of all semimetric spaces is eight; a footnote refers to a note without proofs by J. Haantjes and J. Seidel [Nederl. Akad. Wetensch., Proc. 50, 892–894 = Indagationes Math. 9, 403–405 (1947); these Rev. 9, 299] where the congruence order is announced to be 7. The congruence order reduces to 5 if the semimetric spaces satisfy the following condition: for any three points p_1, p_2, p_3, the triple (p_1, p_2, p_3) is congruently imbeddable in $E_{1,r}$ or at least two of the distances p_ip_j $(i, j=1, 2, 3; i\neq j)$ equal $\pi r/2$. The equilateral subsets of $E_{2,r}$ and $E_{3,r}$ are completely classified; their study is approached by the investigation of two-distance sets in two- and three-dimensional Euclidean spaces. Theorem: the property of being a metric basis for the $E_{2,r}$ is not a congruence invariant unless the basis contains more than five points. All proofs are given in detail; some of them involve numerical evaluations of angles and require a classification of configurations. The matrix technique going back to Menger is still used at the beginning of the paper; in this connection it is interesting to mention that a matrix theorem is derived from the congruence order theorem for $E_{3,r}$. An elementary geometrical proof for the simple property expressed by lemma 4.1, page 89, would be welcome.

C. Y. Pauc (Cape Town).

Di Noi, Salvatore. La continuità della retta e il postulato V° di Euclide. Boll. Un. Mat. Ital. (3) 4, 410–412 (1949).

Fabricius-Bjerre, Fr. Nichteuklidische Fusspunktkurven. Monatsh. Math. 53, 298–301 (1949).

Die Untersuchung von Fusspunktkurven in einer nichteuklidischen Ebene wird unter Benutzung des Cayley'schen Modells auf den euklidischen Fall durch folgenden Satz zurückgeführt. Die nichteuklidische Fusspunktkurve eines Punktes A in Bezug auf eine Kurve k kann durch eine Kollineation in die euklidische Fusspunktkurve des A entsprechenden Punktes in Bezug auf die k entsprechende Kurve übergeführt werden. Klassische Sätze wenn k ein Kegelschnitt ist, werden leicht übertragen, z.B.: Die auf derselben Achse liegenden Brennpunkte F und F_1 eines nichteuklidischen Kegelschnittes k haben dieselbe Fusspunktkurve in Bezug auf k, nämlich einen Kegelschnitt k_1, der k in den auf der Brennpunktsachse liegenden Scheitelpunkten berührt und durch die Berührungspunkte der Tangenten von F und F_1 an den absoluten Kegelschnitt geht. Dasselbe Übertragungsprinzip lässt sich auf den dreidimensionalen Fall anwenden. Beispiele wenn die Fläche eine Fläche zweiter Ordnung oder die Tangentenfläche einer kubischen Kurve ist, werden gegeben. *C. Y. Pauc.*

Nestorovič, N. M. On the equivalence of a hypercircle to an ordinary circle in constructions in the Lobačevskiĭ plane. Doklady Akad. Nauk SSSR (N.S.) 69, 731–734 (1949). (Russian)

All geometrical constructions of the second order in the Lobačevskiĭ plane can be carried out by means of a ruler and an apparatus to draw hypercircles. The author gives a description of such a drawing instrument. The elementary constructions of Steiner are discussed. After that the author solves the two main problems: the constructions of the intersections of a circle with a line and of two circles, when the circles are given by centre and radius. Some derived constructions are also given. This paper, together with his earlier work, shows that each of the instruments (a) an ordinary circle, (b) a horocycle, (c) a hypercircle, together with a ruler, suffice for the solution of second order constructions. *H. A. Lauwerier* (Amsterdam).

v. Sz. Nagy, Gyula. Ein topologischer Satz über endliche geschlossene Kurven in der Ebene. Elemente der Math. 4, 85–86 (1949).

Let there be given two plane curves C and K, both closed and of finite arc length; K is a Jordan curve while C may have a finite number of double points. Suppose finally that K and C have only a finite number of points in common. The natural order on K of those points in which the two curves intersect is a rearrangement of their natural order on C. A study of this rearrangement yields a lower bound for the number of double points of C inside or outside of K. *P. Scherk* (Saskatoon, Sask.).

Monticelli, Edgarda. Determinazione di una classe di curve unicursali il cui arco si esprime a mezzo di un integrale ellittico di prima specie di Legendre. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 107, 107–121 (1949).

For every positive integer n, the author determines rational functions $x_n(s)$, $y_n(s)$ of the real parameter s which define a curve C_n which has arc length proportional to the elliptic integral of the first kind, $F(\varphi, k)=\int_0^\varphi(1-k^2\sin^2\varphi)^{-\frac{1}{2}}d\varphi$, with modulus $k=n^{\frac{1}{2}}(n+1)^{-\frac{1}{2}}$. More general results on the determination of such curves were given by Cayley [An Elementary Treatise on Elliptic Functions, Cambridge, 1876]. The present paper elaborates on a special case of Cayley's results. It should be pointed out that the proof [on pages 115–116] that $\int ds=cF(\varphi, k)$, c=constant, is incomplete. *H. T. Muhly* (Iowa City, Iowa).

Emersleben, Otto. Geometrischer Beweis einer Enveloppeneigenschaft monokonfokaler Ellipsen mit gleich langer grosser Achse. Math. Nachr. 3, 62–70 (1949).

By purely geometrical means the author proves the following proposition. Let P and S be two points in the plane and let a be a line segment. Let E be the set of all ellipses through P with major axis $2a$ and with S as one of the focal points. Then the envelope U of E is the ellipse with P and S as foci and $4a-PS$ as major axis. A construction for the tangent point of U with an arbitrary member of E is given. The author establishes similar results for parabolas and hyperbolas. *W. van der Kulk* (Providence, R. I.).

Sz-Nagy, Gyula. Über die Lemniskatenflächen. Ann. Scuola Norm. Super. Pisa (3) 2 (1948), 39–53 (1950).

The lemniscate surfaces $L(\rho)$ investigated are those with equations of the form

$$F(x, y, z)\equiv\prod_{k=1}^{n}[(x-x_k)^2+(y-y_k)^2+(z-z_k)^2]=\rho^{2n},$$

where ρ is a constant. The points Q_k: (x_k, y_k, z_k) are called the center points of $L(\rho)$. The singular points of $L(\rho)$ are the zeros of the "derivative" of the "distance-polynomial" $F(x, y, z)$ previously studied by the author [Bull. Amer. Math. Soc. 55, 329–342 (1949); these Rev. 10, 702]. As such, the real singular points lie in the convex hull H of center points of $L(\rho)$ and have some other properties of the zeros of the derivative of a polynomial. The author also studies the normals and curvature of the $L(\rho)$ as well as the location of $L(\rho)$ relative to spheres with centers at the centers of $L(\rho)$ and the generalization of various properties of plane lemniscates to the $L(\rho)$. *M. Marden.*

Miheev, V. I. The number of forms of homology of crystals. Doklady Akad. Nauk SSSR (N.S.) 71, 667–670 (1950). (Russian)

The elementary properties of reflections and rotations are treated projectively in such a manner as to facilitate the enumeration of crystal classes. *H. S. M. Coxeter.*

Kolmogorov, N. A. Basic formulas of hyperspherical tetrahedrometry. Učenye Zapiski Moskov. Gos. Univ. 135, Matematika, Tom II, 188–191 (1948). (Russian)

A straightforward elementary development of the trigonometry of the three-sphere. *H. Busemann.*

Thébault, Victor. A note on orthopolar triangles. Amer. Math. Monthly 57, 171–173 (1950).

Goormaghtigh, R. On pairs of triangles. Amer. Math. Monthly 57, 150–153 (1950).

Court, N. A. A special tetrahedron. Amer. Math. Monthly 57, 176–177 (1950).

Continuation of a paper with the same title [same Monthly 56, 312–315 (1949); these Rev. 10, 618.

Droussent, Lucien. On the orthocentroidal circle. Amer. Math. Monthly 57, 169–171 (1950).

Convex Domains, Extremal Problems

***Gericke, Helmuth. Konvexe Körper und Differentialgeometrie im Grossen.** Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 217–230. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10=$2.40.

Chapter headings: Brunn-Minkowskische Theorie, Isoperimetrie; Vierscheitelsatz; Kennzeichnung besonderer Eibereiche; Verschiedene Einzelfragen. *P. Scherk.*

Schmidt, Erhard. Die Brunn-Minkowskische Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und nichteuklidischen Geometrie. II. Math. Nachr. 2, 171–244 (1949).

This review of part II (chapters III and IV) of this paper is based on that of part I [cf. Math. Nachr. 1, 81–157 (1948); these Rev. 10, 471]. Chapter III deals with linear improvements of (1), (3), (5). A discussion of (3) may suffice. Let $I_n(\rho)$ denote the volume of the solid sphere of radius ρ in R_n; $I_n'(\rho)=dI_n(\rho)/d\rho$. Let R^* denote the maximum distance of $\mathfrak{R}^*$ from the axis of our rotational symmetrization, i.e., from the straight line through O perpendicular to E. If $R^*\geqq r$ and if $\mathfrak{R}^{(h)}$ is not empty, then

$$(6)\quad V(\mathfrak{R}^{(h)})\leqq I_n(h-R^*)+\frac{I_n'(h-R^*)}{I_n'(R^*)}(I_n(R^*)-V(\mathfrak{R})).$$

Thus (6) is an improvement of (3) if $R^*>r$ and it reduces to (3) for $R^*=r$. Suppose that $\mathfrak{R}$ possesses an $(n-1)$-dimensional plane intersection of volume at least $I_{n-1}(R)$, where $R>r$ and in the spherical case also $R\leqq\pi/2$. Then $R^*\geqq R$ and (6) will hold with R instead of R^* provided only that $\mathfrak{R}^{(h)}$ is not empty. The author obtains (6) from inequalities used to prove (4).

Most of the last chapter discusses the equality sign in (5). Suppose that all the intersections of $\mathfrak{R}$ with the solid

n-spheres about the points of $\mathfrak{K}$ have positive Lebesgue measures, and suppose in the spherical case that $\overline{\mathfrak{K}}$ is not empty. Then $O(\mathfrak{K})=O(\mathfrak{K}_0)$ implies $\mathfrak{K}=\mathfrak{K}_0$. The last section contains a short proof of the "Spiegel-theorem" in Euclidean R_n, in which this theorem is reduced to the classical Brunn-Minkowski inequality for the volume of a linear combination of two convex bodies. *P. Scherk.*

Dinghas, Alexander. Zur Theorie der konvexen Rotationskörper im n-dimensionalen Raum. Math. Nachr. **2**, 124–140 (1949).

Let $\mathfrak{K}$ be a convex body in n-space formed by revolving a plane curve about a straight line l, and let $\mathfrak{K}_h$ denote the union of the spheres of radius $h>0$ about the points of $\mathfrak{K}$. Thus the volume of $\mathfrak{K}_h$ is a polynomial $V(\mathfrak{K}_h)=\sum_0^n\binom{n}{\nu}V_\nu h^\nu$ in h. Let d denote the maximum distance of any point of $\mathfrak{K}$ from l. In a previous paper [Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1939**, no. 17; these Rev. **2**, 262] the author proved

$$V_k-2V_{k+1}d+V_{k+2}d^2\leqq 0, \qquad k=0,1,\cdots,n-2. \tag{1}$$

He now proves two sets of inequalities which include (1) and

$$V_k-3V_{k+1}d+3V_{k+2}d^2-V_{k+3}d^3\geqq -S_kd^{n-k}, \qquad k=0,1,\cdots,n-3, \tag{2}$$

as special cases. Here

$$S_k=\pi^{-1}W_{n-1}\int_0^\pi(1-\sin\theta)^2(1+\sin\theta)^{-1}\sin^k\theta\,d\theta,$$

where W_{n-1} is the volume of the unit sphere in $(n-1)$-space. [For the case $n=3$ cf. Hadwiger, same vol., 114–123 (1949); these Rev. **11**, 127.] The cases of equality are discussed for each relation. These results are obtained as corollaries of identities which express the left hand terms of (1), (2), etc., by means of integrals. *P. Scherk* (Saskatoon, Sask.).

Pogorelov, A. V. A theorem of uniqueness for convex surfaces. Mat. Sbornik N.S. **26(68)**, 147–152 (1950). (Russian)

A convex surface is a connected open subset of the boundary of a convex body in E^3. Let F_1 and F_2 be two nonplane convex surfaces, and let a mapping of the points of F_1 on the points of F_2 and of the supporting planes of F_1 on the supporting planes of F_2 exist with the following properties: the intrinsic distances of corresponding pairs of points are equal; if π_1 is a supporting plane of F_1 at p_1 and π_2 the corresponding supporting plane of F_2, then π_2 contains the image of p_1; the angles between the exterior normals of corresponding pairs of supporting planes of F_1 and F_2 are equal. Then a motion of E^3 exists (not necessarily orientation-preserving) which carries F_1 into F_2. *H. Busemann.*

Jackson, S. B. Geodesic vertices on surfaces of constant curvature. Amer. J. Math. **72**, 161–186 (1950).

Given a curve C of class C'' on a surface of constant curvature. At a "geodesic vertex" of C, the geodesic curvature of C has a relative extremum. An (isolated) geodesic vertex can be characterized geometrically by the fact that its osculating geodesic circle supports C locally. The author studies "monotone arcs" without geodesic vertices and closed curves with exactly two geodesic vertices in a simply connected subregion S of the surface [for the plane case cf. Bull. Amer. Math. Soc. **50**, 564–578 (1944); these Rev. **6**, 100]. He gives another proof of the four-vertex theorem on S [cf. same J. **67**, 563–582 (1945); these Rev. **7**, 259] and extends to S the classical sufficient conditions for a simple closed curve to have exactly four or at least $2n$ geodesic vertices [cf., e.g., the first reference]. *P. Scherk* (Saskatoon, Sask.).

Algebraic Geometry

***Semple, J. G., and Roth, L. Introduction to Algebraic Geometry.** Oxford, at the Clarendon Press, 1949. xvi+446 pp. $7.50.

The ground covered in this book is very extensive: almost every topic of the classical algebro-geometric theory of algebraic curves, surfaces and varieties is discussed, or at least briefly mentioned, in the course of the exposition. Since it is obviously impossible to write a treatise in one volume on the whole of algebraic geometry (even if the transcendental and topological theories are excluded), one would be tempted to conclude a priori that the present book must be something in the nature of an encyclopedia article. This, however, is not the case. While the book is much less than a treatise in which the subject matter is not only presented but also developed step by step with complete rigor, it is also much more than a formal report of results. For one thing, the authors have included in the text a very large number of special but important examples (special curves, surfaces, special transformations), and these examples are discussed in great detail. It is in these examples that the ideas and methods of the general theories are put to work on concrete situations. This wealth of experimental material will be welcome even by the specialist, but it will be really invaluable to the beginner who wishes to acquire a geometric insight and develop a geometric technique. In the second place, the theorems which belong to the general theoretical topics of the book are not merely stated. There is a definite attempt to prove them or at least to justify them to the reader.

This raises of course the delicate question of what constitutes a mathematical proof. The "proofs" which are to be found in the present book fall roughly into three classes: (1) proofs; (2) admittedly incomplete proofs; (3) plausibility arguments. The proofs of the first class are to be found (as a rule) in the elementary phases of the subject matter where no foundational difficulties are present (as, for instance, in the projective theory of plane algebraic curves and, to a lesser extent, of surfaces in three-space). In the proofs of the second class the authors stop at a certain point and explicitly state that the rest of the proof is beyond the scope of the book, a perfectly legitimate procedure. Occurrences of this nature are frequent in the book, especially in cases when really advanced questions are taken up for discussion. But by far the great majority of the proofs belong to the class of plausibility arguments. These arguments are not without value, and they are presented with that delightful and refreshing informality that is characteristic of most expositions in the field of classical algebraic geometry. The only criticism that we wish to offer is this: it is unfortunate that these arguments are, as a rule and without further ado, presented to the reader as actual proofs, instead of being relegated to the proofs of the second class (where they really belong). A young reader who has been exposed to the standards of rigor in modern mathematics but who has not yet mastered the foundational aspects of modern algebraic geometry will easily be bewildered and discouraged by his inability to follow these

proofs intelligently. The authors state in the preface that their book requires "no more background than the usual honours degree courses in projective geometry and algebra." It seems to the reviewer that the background which is actually required is either much less or much more than that. The student who reads this book must be either sufficiently immature not to be in position to question the validity of the proofs or he must be sufficiently advanced in algebraic geometry to know how the arguments of the authors can be made rigorous.

A brief description of the contents of the various chapters will now suffice in order to indicate the scope of the book. Chapter I (introduction) deals with algebraic manifolds, their intersections and with algebraic correspondences. These basic foundational topics are treated very sketchily and informally, and the authors themselves advise the reader "to read the chapter very lightly . . . and to pass quickly to chapter II" (Plane curves) in which multiple points, the polar curves and the Hessian of a plane curve are discussed, with applications to rational and elliptic curves. Chapter III (The quadratic transformations) deals with the concept of infinitely near points in the plane, intersection multiplicities and the resolution of the singularities of plane algebraic curves by quadratic transformations. Chapter IV (Rational transformations) deals with a variety of topics such as algebraic correspondences (m, n) on the line and in the plane, the rule of Zeuthen and its applications to the theorem of Bezout and the Plücker equations, the invariance of the genus of a curve under birational transformations, and so on. Chapters V and VI (Systems of plane curves. Linear systems of curves and their projective models) include in addition to the standard general properties of linear systems of curves in the plane also the study of the plane representation of rational surfaces and generalities on the Jacobian systems.

The contents of these two chapters are then specialized and applied in chapter VII (Special rational surfaces and plane Cremona transformations). This chapter, and also chapter VIII (Linear systems of surfaces, rational manifolds, and higher Cremona transformations), have a rich geometric content and a generous supply of special but highly interesting examples. A careful reading of these two chapters cannot be too highly recommended. Chapter IX (Projective characters of curves and surfaces) develops the general projective numerical invariants of surfaces, with applications to ruled surfaces and surfaces that are complete intersections. Chapter X (The geometry of line systems) gives the authors another excellent opportunity of studying a number of special varieties, those related to the representation of lines in S_3, S_4 and S_5. Chapter XI (Some problems in enumerative geometry) develops the Schubert calculus, studies enumerative problems on quadric surfaces and concludes with general remarks on the specialization principle in enumerative geometry.

Chapter XII (Geometry on a curve) contains an algebro-geometric treatment of the birational theory of algebraic curves, including the proof of the theorem of Riemann-Roch and a section on correspondences with valence. Following Enriques, the canonical series is defined by means of the Jacobian series of a g_n^1. The proof of the theorem of Riemann-Roch is based on Noether's reduction theorem. The book ends with the long and important chapter XIII (Geometry on an algebraic surface) which deals with the most advanced phase of algebraic geometry: the birational theory of algebraic surfaces. Naturally, this final chapter only introduces some of the basic ideas of the theory of surfaces, and for the proofs of the deeper results (such as the generalized theorem of Riemann-Roch, the elimination of exceptional curves, conditions of rationality of an algebraic surface, etc.) the reader is referred to original memoirs and other sources. The authors stop short of that part of the theory of surfaces which goes beyond linear equivalence of curves on a surface and which therefore deals with continuous systems of curves (the variety of Picard, Severi's theory of the base, etc.). *O. Zariski.*

***Gröbner, Wolfgang. Moderne algebraische Geometrie. Die idealtheoretischen Grundlagen.** Springer-Verlag, Wien und Innsbruck, 1949. xii+212 pp.

The past decade has seen the publication of various books on algebraic geometry whose object has been the presentation of the foundations of the subject on the rigorous basis made possible by modern methods. In addition to the books of van der Waerden, Hodge and Pedoe, and Weil, there is the forthcoming volume of Zariski. The present book has (apart from the elementary portions) a fairly small intersection with these others. The content of this book is closely related to Macaulay's classical "The Algebraic Theory of Modular Systems" [Cambridge University Press, 1916], much of the material of which is here presented in modern, rigorous form. Of the progress in algebraic geometry made in the last ten years there is no mention (except for the concept of normal variety, as noted below).

The first chapter deals with polynomial rings and power series rings, and with algebraic and transcendental field extensions. The discussion of the ideal theory is continued in the second chapter. The zeros of ideals and algebraic manifolds are here introduced and are studied in part by means of elimination theory. Chapter III deals with the dimension theory of polynomial ideals and includes such topics as mixed and unmixed ideals, ideals of the principal class, polars and tangents. The concepts of integral dependence and integral closure are developed and the principal ideal theorem proved. One is surprised to find no mention of Zariski in connection with the notion of normal variety (here called "supernormal") and the proof of the completeness of the system of hyperplane sections of such a variety. Chapter IV deals with the Hilbert function of an ideal and the order of an algebraic variety. Bézout's theorem is proved in the special case where one of the varieties is represented by a perfect ideal and the other by an ideal of the principal class. The final chapter concerns the theory of the syzygetic chain of a homogeneous ideal and its applications.

Only a very slight knowledge of algebra and n-dimensional projective geometry is required for reading this book, so that it is largely self-contained. The exposition, however, may present some difficulties to the inexperienced reader. There is, for example, frequently some vagueness as to what the ground field K is in a given discussion. On page 3 the author stipulates that K shall always be the complex field, unless the contrary is stated, but he does not take this convention very seriously since throughout most of the book the context shows that K is not considered to be algebraically closed.

A more serious objection is the author's concept of the algebraic manifold of a polynomial ideal A, as contrasted with the zero manifold (Nullstellengebilde) of A. The latter is simply the set of all zeros of A. The former consists of the zeros of A together with certain "infinitely near" points or manifolds. The term "infinitely near" is not defined; the

author merely illustrates its meaning heuristically for several explicit zero-dimensional primary polynomial ideals in two variables, and he suggests that the extension to polynomial ideals in many variables will present no difficulties in principle. [But it must be pointed out that a rigorous treatment of the subject, even for polynomials in two variables, is by no means elementary, as is shown by Zariski's paper on infinitely near points, Amer. J. Math. 60, 151–204 (1938).] The author subsequently uses algebraic manifolds as if they were in one-to-one order-reversing correspondence with ideals; presumably, then, they might just as well have been defined in this way. [But here again there is a difficulty, since even in the classical theory (for polynomials in two variables) there is no such one-to-one correspondence.]

Intersection multiplicity having been defined in terms of length of an ideal, Bézout's theorem can be proved only in a special case, as stated above. The author remarks that the intersection theories of Severi and van der Waerden permit a general proof of Bézout's theorem, but he is very critical of these theories, partly on the ground that they involve continuity considerations and hence are not applicable to general ground fields. In point of fact, however, van der Waerden's multiplicity definition does not involve continuity at all. Of the intersection theories of Chevalley and of Weil there is no mention. *I. S. Cohen.*

Koizumi, Shoji. On the differential forms of the first kind on algebraic varieties. J. Math. Soc. Japan 1, 273–280 (1949).

In this paper the author settles some of the problems on abstract differential forms which were raised by A. Weil [Foundations of Algebraic Geometry, Amer. Math. Soc. Colloquium Publ., v. 29, New York, 1946; these Rev. 9, 303; see in particular pp. 245–247]. The methods employed are those of Weil. First it is shown that a linear differential form dz of a separable extension $k(x)$ of $k(u)$ can be expressed in the expected manner as $dz=\sum_{i=1}^{n} f_i du_i$ with $f_i \epsilon k(x)$ if $k(u)$ has transcendence degree n; second, the differential forms dx_i can be expressed in the classical fashion in terms of the differentials dt_i belonging to the uniformizing variables t_i of a simple point of a variety U; and third, the passage to complete abstract Varieties (in the sense of Weil) is made. Subsequently all Varieties are complete and without singularities. A linear differential form (or also more generally a form of any degree) ω on a Variety $\mathbf{U}$ is called of the first kind if, for every simple Point $\mathbf{P}'$ of $\mathbf{U}$, the coefficients $f(\mathbf{P}')$ of $\omega=\sum_{i=1}^{n} f_i(\mathbf{P}')dt_i$ lie in the specialization ring of $\mathbf{P}'$. Then the principal result holds that every linear differential form ω of the first kind on a Variety is also finite at every simple Point of any birationally equivalent Variety. Furthermore the author proves that (i) such ω's induce differential forms of the first kind on any complete Subvariety $\mathbf{V}$ without multiple points of $\mathbf{U}$, (ii) every linear differential form of the first kind on the product $U\times U_1$ of the Varieties $\mathbf{U}$ and $\mathbf{U}_1$ is the sum of such on the components, and (iii) the dimension of the space of differential forms of the first kind is invariant under extension of the field of definition of the Variety. *O. F. G. Schilling.*

Segre, Beniamino. Le rette delle superficie cubiche nei corpi commutativi. Boll. Un. Mat. Ital. (3) 4, 223–228 (1949).

Let F be a cubic surface defined over a field γ (not of characteristic 2), and without singularities over any algebraic extension of γ. Proofs are sketched of the following results. If γ is algebraically closed then F contains 27 lines over γ whose incidence relations are the same as when γ is the complex field. If γ is not algebraically closed ten different cases can arise: the possible numbers of lines and the incidence relations being as in the case of the rational field. These give rise by projection to eleven possible configurations of bitangents to a plane nonsingular quartic. It is shown by examples that the case of characteristic two is quite exceptional. *D. B. Scott* (London).

Marchionna, Ermanno. Un complemento del teorema dell' $Af+B\varphi$. Boll. Un. Mat. Ital. (3) 4, 368–370 (1949).

The familiar $Af+B\varphi$ theorem states that if a plane curve $\theta=0$ has multiplicity $r\geqq i+l-1$ in every point which is respectively i-ple, l-ple for curves $f=0$, $\varphi=0$, then $\theta\equiv Af+B\varphi$. The present paper extends this result to the case where $r>i+l-m-n+g$, where m, n, g are the orders of f, φ, θ. This is a weaker condition if $g<m+n-2$. *P. Du Val.*

Chisini, Oscar. Geometria numerativa. Rend. Sem. Mat. Fis. Milano 19 (1948), 1–16 (1949).

A short introduction to enumerative geometry which lays special emphasis on the topological aspect of the principles of Chasles and Cayley-Brill. *D. Pedoe* (London).

Chisini, Oscar. Sulla identità birazionale di due funzioni algebriche di più variabili, dotate di una medesima varietà di diramazione. Ist. Lombardo Sci. Lett. Rend Cl. Sci. Mat. Nat. (3) 11(80) (1947), 3–6 (1949).

The author shows, by taking generic hyperplane sections, how the problem of determining whether certain algebraic varieties of $r>2$ dimensions with the same ramification variety are birationally equivalent can be reduced to the corresponding problem for two variables.
O. F. G. Schilling (Chicago, Ill.).

Vaccaro, Giuseppe. Sugli spazi lineari luoghi di flessi di una ipersuperficie algebrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7, 61–66 (1949).

L'auteur étend un résultat de Bompiani [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 497–510 (1941); ces Rev. 8, 227] relatif aux surfaces, au cas des hypersurfaces. Considérant une V_{r-1}^{n} qui contient une droite lieu d'inflexions, il montre que sur cette droite il existe $n-1$ points doubles bihyperplanaires et réciproquement. Si sur l'hypersurface il existe un S_k lieu d'inflexions, une de ses droites jouit de la propriété précédente d'où résulte que le S_k contient une V_{k-1}^{n-1} lieu de points doubles bihyperplanaires. Ceci s'étend immédiatement. La condition nécessaire et suffisante pour qu'un S_k soit lieu de points s-ples à tangentes $(s+2)$-ples sur une V_{r-1}^{n} est qu'il existe sur S_k une V_{k-1}^{n-1} lieu de points $(s+1)$-ples pour V_{r-1}^{n} avec cône osculateur décomposé en un cône d'ordre s osculateur en tous les points de S_k et un hyperplan. *B. d'Orgeval* (Grenoble).

Zammataro, Nicola. Sulle curve algebriche triple a gruppo di monodromia totale. Matematiche, Catania 4, 77–82 (1949).

L'auteur considère la courbe triple:

$$f(z;u)=u^3+3a(z)u+2b(z)=0$$

dont le groupe de monodromie est le groupe total des substitutions sur trois éléments. Il y associe la résolvante du deuxième degré $\varphi(y,u)$ de déterminant D. Les diramations de φ correspondent aux racines de D de multiplicité impaire à l'exclusion de celles de multiplicité paire. Les diramations de f correspondent soit à des racines de D de multi-

plicité impaire et les substitutions associées n'agissent que sur deux déterminations de u; soit à des racines de D de multiplicité paire, racines également de a et de b, la multiplicité pour b étant multiple de 3, la substitution associée portant sur les trois déterminations. Si n_k est le nombre de diramation (u_i, u_j) $(i \neq j \neq k)$, m_1 et m_2 celui des cycles (1, 2, 3) et (1, 3, 2), le genre de la Riemannienne de u est $p = \sum_1^3 \frac{1}{3} n_i + m_1 + m_2 - 2$. *B. d'Orgeval* (Grenoble).

†Ramser, Hans. Die Diskriminantenhyperfläche von quadratischen Formen. Thesis, Eidgenössische Technische Hochschule in Zürich, 1949. 51 pp.

Quadric loci of S_r (r-dimensional space) are treated as points of $S_{N(r)}$ $(N(r) = \frac{1}{2}r(r+3))$ and the varieties V_d of dimension $d = N(r) - \binom{s+1}{2}$ considered whose points represent s-ply specialized quadrics (cones with $(s-1)$-dimensional vertex). The case $s = r-1$, $d = 2r$, corresponds to pairs of hyperplanes; $s = r$, $d = r$, to double hyperplanes. The case $s = 1$ gives the "discriminant hypersurface," $D^{r+1}_{N(r)-1}$, represented by the vanishing of a symmetrical determinant of order $r+1$ with linear elements; higher values of s, varieties represented by the vanishing of all minors of order $r-s+2$ in this, which is a locus of s-ple points on $D^{r+1}_{N(r)-1}$. In particular, $s = r$ gives the "Veronese figure" V_r^r, and $s = r-1$ a variety V_{2r} generated by the chords, and also by the tangent S_r's, of V_r^r. A formula is given for the order of each of these. For $r = 2$ we have the Veronese surface and the cubic generated by its chords, for $r = 3$ we have D_9^4 in S_9, V_6^{10} which is double in this and V_3^9 which is triple on D_9^4 and quadruple on V_6^{10}. Hypersurfaces of order r in $S_{N(r)}$ which touch $D^{r+1}_{N(r)-1}$ at all common points represent families of quadrics touching a fixed hyperplane in S_r. Some geometrical properties of V_r^r are obtained, and symmetroids in S_2, S_3, S_4 of orders 3, 4, 5 are studied as sections of D_4^3, D_9^4, D_{12}^5. *P. Du Val* (Athens, Ga.).

Godeaux, Lucien. Sur les points de diramation isolés des surfaces multiples. VI. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 828–833 (1949).

Godeaux, Lucien. Sur les points de diramation isolés des surfaces multiples. VII. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 834–840 (1949).

On trouve dans la première de ces deux notes encore un exemple de structure d'un point de diramation isolé d'une surface algébrique multiple [voir même Bull. Cl. Sci. (5) **35**, 15–30, 270–284, 285–292, 532–541, 636–641 (1949); ces Rev. **10**, 735; **11**, 205, 393]. Il s'agit ici de la surface multiple Φ qui représente une involution d'ordre $p = 61$ (et ayant $\alpha = 37$) sur une surface F. Le point de diramation considéré A' a pour Φ la multiplicité 8; son cône tangent se décompose en trois plans et un cône rationnel du 5ième ordre; de manière que la singularité de Φ au point A' équivaut à une droite, une C^3 rationnelle et deux autres droites; ces lignes, prises dans cet ordre, ont les degrés virtuels -2; -7; -3; -2 et chacune d'elles rencontre en un point seulement la précédente et la suivante.

Dans la note suivante, qui est la dernière de toute la recherche, l'auteur revient sur le cas général d'une involution I d'ordre premier $p > 2$ existant sur une surface algébrique F, et ayant un point uni A, auquel correspond sur une surface Φ image de I un point de diramation isolé A'. Il suppose toujours que I, dans le domaine de A, soit engendrée par l'homographie $\lambda' : \mu' = \lambda : \epsilon^{\alpha-1}\mu$, où $1 \leqq \alpha \leqq p$, et ϵ est une racine primitive de l'ordre p de 1. Il démontre que le cône tangent à la surface Φ au point A' se décompose en quatre cônes rationnels au plus, pourvu que l'on suppose que ce cône possède seulement des génératrices doubles. Les trois cas où le cône se décompose en deux, ou en trois, ou en quatre parties, peuvent être caractérisés à l'aide des deux nombres entiers et positifs λ, μ qui rendent $\lambda + \mu$ minimum et tels que $\lambda + \alpha\mu \equiv 0$, $\mu + \beta\lambda \equiv 0 \pmod{p}$; où $\alpha\beta \equiv 1 \pmod{p}$ et $1 \leqq \beta \leqq p$. *E. G. Togliatti* (Gênes).

Franchetta, Alfredo. Sui sistemi pluricanonici di una superficie algebrica. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 423–440 (1949).

A reducible curve on an algebraic surface is called "virtually connected" if, however it is divided into two parts, the virtual intersection number of these is positive. (If no constituent is multiple, this is the ordinary sense of "connected.") Enriques [Rivista Acad. Ci. Madrid **40**, 149–159 (1946); these Rev. **9**, 373] proved that curves of the canonical system on a surface satisfying $p_g = p_a > 0$, $p^{(1)} > 0$ are irreducible or at least virtually connected; that the adjoint system of a virtually connected curve on a regular surface is regular; and that if the canonical system has no multiple fixed parts, the bicanonical system is irreducible. The present note generalizes these results to irregular surfaces as follows. On a surface satisfying $p_a > 0$, $p^{(1)} > 0$, (i) the canonical system is irreducible or virtually connected; (ii) the pluricanonical systems are regular; (iii) the bicanonical system can have no fixed part except possibly rational curves of virtual grade less than -2 (it is not known whether this exception can actually arise); (iv) the tricanonical and higher pluricanonical systems are irreducible.

P. Du Val (Athens, Ga.).

Jongmans, F. Le problème des séries spéciales d'une courbe algébrique. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 1027–1041 (1949).

Remarking that among the unsolved problems in the geometry on an algebraic curve is that relating to the distribution of special linear series, the author considers various possibilities, which cannot be enumerated here. His aim is to prove some simple theorems which will throw some light on the subject. *D. Pedoe* (London).

Jongmans, F. Le problème des séries spéciales d'une courbe algébrique. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 1113–1124 (1949).

Continuation of the investigation described above. Much of the work deals with curves which possess two g_l^1, or a g_l^1 and a g_k^1 $(2 < l < k)$. *D. Pedoe* (London).

Nollet, Louis. Généralités sur les surfaces algébriques dont le système canonique est composé au moyen d'un faisceau. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 1016–1026 (1949).

The author considers a surface F of genus $p_g > 1$ and irregularity q, the general curve of whose canonical system $|K|$ consists of a fixed part f (of virtual grade and genus v, π) together with $m > 1$ curves variable in a pencil $\{E\}$; the curves E are themselves of genus p, and form a one-dimensional family of genus x; they have n base points, and meet f in k points. The surface is assumed to be in a form without singular points or exceptional curves, so that n and the virtual genus $p^{(1)}$ of $|K|$ are quite definite. In the familiar case $p = p^{(1)} = 1$, $n = 0$, all pluricanonical systems are likewise compounded with $\{E\}$. If on the other hand either $p > 1$ or $p^{(1)} > 1$ holds, so does the other, and the variable part of every pluricanonical system is irreducible. If $\{E\}$

is mapped on a curve of genus x, the sets which form curves of $|K|$ correspond to a complete nonspecial linear series, whence $m=p_g+x-1$. If $n=0$ both p_g and $p^{(1)}$ can be arbitrarily large, but $p\leqq 10$, and if $x>0$, $p\leqq 5$. If $n>0$, then $x=0$, $n\leqq 9$, $p\leqq 15$, $p_g\leqq 12$, $p^{(1)}\leqq 130$, $q\leqq 4$. If either $n=9$ or $p=15$ holds so does the other, and further $q=0$, $m=2$, $p_g=3$, $k=1$, $v=\pi-2$, $\pi=0$ or 1, $p^{(1)}=39+\pi$. If $p_g=12$, $p^{(1)}=122$. *P. Du Val* (Athens, Ga.).

Differential Geometry

*Haupt, Otto. **Theorie der geometrischen Ordnungen.** Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 197–215. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

This is a detailed report on work that is due mostly to Haupt and his students and that is still partly unpublished. The topics dealt with are indicated by the table of contents. A. Fragestellungen. B. Lokale Probleme. 1. Untersuchungen über singuläre Punkte auf Bogen im E_n. [In this section their classification by means of signed permutations is discussed; cf. Denk and the author, J. Reine Angew. Math. 183, 69–91 (1941); these Rev. 4, 113.] 2. Bestimmung der elementaren und nichtelementaren Singularitäten (von beschränktem Ordnungswert) auf Bogen im E_3 und E_4. 3. Bestimmung ordnungshomogener Teile. [This section includes a discussion of Nöbeling's papers on the topological structure of sets of finite order.] 4. Direkte Infinitesimalgeometrie. 5. Über einige affingeometrische Oval-Sätze in der direkten Infinitesimalgeometrie. [This includes generalizations of Dupin's indicatrix from the point of view of direct differential geometry.] C. Globale Probleme. 1. Limessätze bei geometrischen Ordnungen. 2. Ordnungsminimale Kontinua im projektiven Raum R_n. 3. Windungsmonotonie der Bogen vom Ordnungswert n im E_n. 4. Böhmersche und verwandte Ovalsätze. D. Anhang.

P. Scherk (Saskatoon, Sask.).

Aleksandrov, A. D. Quasigeodesics on manifolds which are homeomorphic to a sphere. Doklady Akad. Nauk SSSR (N.S.) 70, 557–560 (1950). (Russian)

A quasigeodesic on a surface with bounded curvature is a curve whose total right and left geodesic curvatures have the same sign. (Every geodesic is a quasigeodesic, but the converse is not always true.) By approximation with regular surfaces the following theorems are derived from the corresponding theorem of Lusternik and Schnirelmann and of M. Morse for the regular case: On every surface S of bounded curvature homeomorphic to a sphere there are three simple closed quasigeodesics; more precisely: there are three such quasigeodesics with different lengths, or there is a whole family covering S. There are infinitely many quasigeodesics passing through two given points of S. There are infinitely many quasigeodesic one-gons with a given vertex.

H. Busemann (Los Angeles, Calif.).

*Bol, Gerrit. **Projektive Differentialgeometrie. I. Teil.** Vandenhoeck & Ruprecht, Göttingen, 1950. vii+365 pp. 17.80 DM; bound, 20 DM.

This first volume deals substantially with the theory of curves and tapes (Kurvenstreifen). The author uses a device which in his hands is a very useful tool for geometric investigation and distinguishes the book from other books about projective differential geometry: normalization of projective coordinates is avoided and the parameter transformation is combined with a change of factor of proportionality in such a way that a certain basic determinant remains invariant with respect to this combined transformation.

In the first chapter (devoted to plane curves) the determinant in question is $(\mathfrak{x}, \mathfrak{x}_1, \mathfrak{x}_{11})$, supposed constant and different from zero. Then a point frame $\mathfrak{x}$, $\mathfrak{i}=\mathfrak{x}_1$, $\mathfrak{y}$ for our curve $C(\mathfrak{x}(t))$ is easily found as well as its corresponding Frenet formulae with two functions a, b. The locus of $\mathfrak{y}$ under the combined transformations mentioned above is the osculating conic section of C at $\mathfrak{x}$ (and $\mathfrak{i}\mathfrak{y}$ is its tangent at $\mathfrak{y}$). In particular, $b=0$ characterizes a sextactic point. If $b\neq 0$ along C then $s=\int b^{\frac{1}{3}}dt$ is its projective arc and $a=k(s)$ its projective curvature. The condition $k=$constant characterizes the W-curves (e.g., the curves which admit a one-parameter group of projective transformations). As is usual in projective geometry, any theorem has its counterpart in a dual theorem. Geometry in the large is represented by theorems about the number of sextactic points. (Example. Let C be closed, without points of inflection, intersected by any straight line at most in two points and having the same osculating conic sections at two different points. Then it has at least ten sextactic points.)

The second chapter is more or less an introduction to projective geometry in space. Besides elementary theorems about line geometry (ruled surfaces, linear congruences, linear complexes, in particular, connected with algebraic curves of the third order) the Lie and Darboux quadrics as well as Darboux and Segre directions are introduced. Together with the elements of a general congruence the W-congruences as well as the Laplace transforms are discussed. The chapter ends with an elaborate study of different kinds of contact of curves and surfaces.

The third chapter deals with skew curves. The device mentioned before is very efficient here inasmuch as it enables the author to enrich considerably the projective theory of skew curves. Thus for instance the point $\mathfrak{z}$ of the frame $\mathfrak{x}$, $\mathfrak{i}=\mathfrak{x}'$, $\mathfrak{y}$, $\mathfrak{z}$ of $C(\mathfrak{x})$ describes (by the combined transformations described above) an algebraic curve C_3 which has five point contact with C at $\mathfrak{x}$ and leads in a natural way to the Fubini C_3's whose adjoint linear complex is the osculating complex of C at $\mathfrak{x}$. It is impossible even to mention in a short review all the interesting topics of this chapter. As an example (which by no means covers all the topics) we mention the role played by the functions a, b, c of the Frenet formulae for the frame $\mathfrak{x}$, $\mathfrak{i}$, $\mathfrak{y}$, $\mathfrak{z}$. Thus for instance $b=0$ characterizes the curves of a linear complex, $a=0$ ($c=0$) characterizes the curves for which the lines $\mathfrak{x}\mathfrak{z}$ ($\mathfrak{i}\mathfrak{z}$) are rulings of a developable surface. Moreover $c=0$ characterizes curves for which the surface of all C_3 and the surface of their osculating planes coincide. If and only if for a special choice of parameter the functions a, b, c are constant, C is a W-curve. For other results of this chapter not mentioned in this review, cf. the author's paper [Abh. Math. Sem. Univ. Hamburg 16, 1–28 (1949); these Rev. 11, 437].

In the fourth chapter, dealing with tapes, the author chiefly develops in detail his previous results [Arch. Math. 1, 192–199 (1948); 362–370, 371–376 (1949); these Rev. 10, 402; 11, 52, 53].

Throughout the book each section is followed by many examples or new theorems connected with the topics dealt with in the corresponding section. The bibliography at the end of the book is a continuation of the bibliography of the

book of Fubini and Čech [Introduction à la géométrie projective différentielle des surfaces, Gauthier-Villars, Paris, 1931]. The book is written very clearly and requires little previous knowledge. The constant emphasis on geometric applications makes it very attractive. *V. Hlavatý.*

Sorace, O. Iperosculazione del terzo ordine di una curva data in un suo punto. Boll. Accad. Gioenia Sci. Nat. Catania (4) no. 1, 54–58 (1948).

Computation of a curve which has a contact of order three with a given curve at their common point.

V. Hlavatý (Bloomington, Ind.).

Jarník, Vojtěch. Sur le cercle de courbure. Časopis Pěst. Mat. Fys. **74**, D37–D51 (1949). (Czech. French summary)

The object of this paper is to give students a number of illustrations and exercises in the use of limits of functions of several variables. If $y=f(x)$ represents a continuous curve, P_i the point $(x_i, f(x_i))$, denote by $K(x_1, x_2, x_3)$ the circle passing through P_1, P_2, P_3, by $K(x_1, x_1, x_2)$ the circle passing through P_2 touching the curve at P_1, and by $K(x_1, x_2/n)$ the circle passing through a fixed point P_0 having its centre at the intersection of the normals at P_1, P_2. The circle of curvature at P_0 may be defined by either of the limiting positions of $K(x_1, x_2, x_3)$, $K(x_0, x_2, x_3)$, $K(x_1, x_1, x_2)$, $K(x_1, x_1, x_0)$, $K(x_0, x_0, x_1)$, $K(x_1, x_2/n)$, $K(x_0, x_1/n)$ as x_1, x_2, x_3 tend to x_0. Necessary and sufficient conditions are established for these limiting positions to exist. *F. A. Behrend* (Melbourne).

Boaga, Giovanni. Sulla rappresentazione di Weingarten della sfera sul piano. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 402–407 (1949).

The author calls the orthogonal projection T of a sphere S of radius R upon a tangent plane π (which may be followed by a magnification with respect to the point of contact O of S and π) the representation of Weingarten of S upon π. This is a special case of a corresponding problem studied by Weingarten in general surface theory. The mapping T is studied in detail. If s denotes the arc length along a meridian of S from O and if ψ is the angle between a meridian and a curve on S, then $n^2=(d\sigma'/d\sigma)^2=\cos^2\psi\cos^2(s/R)+\sin^2\psi$, where $d\sigma'$ and $d\sigma$ are the linear elements on π and S. The modulus of surface deformation is $m=dA'/dA=\cos(s/R)$, where dA' and dA are the corresponding differentials of area on π and S. If ψ and ψ' denote the corresponding angles on S and π, then $\tan\psi'=\sec(s/R)\tan\psi$. The angular deformation $\chi=\psi'-\psi$ is given by

$$\tan\chi=(1-\cos(s/R))\tan\psi/(\cos(s/R)+\tan^2\psi).$$

Then $\tan\chi_{\max}=\pm\sin^2(\frac{1}{2}s/R)(\sec(s/R))^{\frac{1}{2}}$. These formulas are applied to geography. Finally conformal representations of S upon π are discussed. *J. DeCicco* (Chicago, Ill.).

Efimov, N. V. Research on the deformation of surfaces containing points with zero Gaussian curvature. Amer. Math. Soc. Translation no. **15**, 68 pp. (1950).

Translated from Mat. Sbornik N.S. **23(65)**, 89–125 (1948); these Rev. **10**, 265.

Kovancov, N. I. On a canonical bundle. Mat. Sbornik N.S. **26(68)**, 153–160 (1950). (Russian)

The projective-differential properties of surfaces are usually expressed by means of certain canonical lines such as the directrix of Wilczynski, the normal of Fubini, etc. The author considers the general problem by inquiring into the form of $f(\beta, \gamma, du, dv)$ which remains invariant under the transformation $u=u(\bar{u})$, $v=v(\bar{v})$ which also leaves Fubini's metric $(\beta du^3+\gamma dv^3)/dudv$ invariant. The general solution of this problem is obtained in terms of a number of parameters and the canonical lines are given by a form $\varphi=\beta^\alpha\gamma^t du^{2\alpha-t}dv^{2t-\alpha}+\beta^t\gamma^\alpha du^{2t-\alpha}dv^{2\alpha-t}$. Thus $\alpha=t=1$ gives the normal of Fubini, $\alpha=1$, $t=0$ gives the line of Cartan, $\alpha=2$, $t=1$, the directrix of Wilczynski.

M. S. Knebelman (Pullman, Wash.).

Longo, Carmelo. Sulle trasformazioni tra piani che mutano un determinato fascio di rette in un fascio di rette. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 66–72 (1949).

Study of point-transformations between two projective planes such that one pencil of lines is transformed into a pencil of lines. The following cases are examined: (1) two of the inflexional lines through an arbitrary point coincide with the line of the pencil through that point; (2) two of the inflexional lines coincide in a line different from the preceding one; (3) the two inflexional lines through a point different from the join of that point with the center of the pencil determine an involution (when the point varies) on a fixed line of the pencil; (4) the two mentioned inflexional lines pass through two fixed points collinear with the center of the pencil. The finite equations of these transformations are given, using an interpretation in ordinary space. In case of a Cremona transformation, the preceding conditions are only satisfied by De Jonquières transformations; these are completely characterized in the four cases in connection with the configuration of their fundamental points.

E. Bompiani (Rome).

Sangermano, Cosimo. Le trasformazioni puntuali fra due piani in una coppia a Jacobiano nullo di caratteristica zero. Boll. Un. Mat. Ital. (3) **4**, 260–267 (1949).

Study of point-transformation between projective planes in the neighborhoods of a pair of corresponding points where the Jacobian has rank zero. Intrinsic reference systems are determined by the neighborhoods of the fourth order; a canonical form is given, together with covariant lines and curves associated with the neighborhoods of the third and fourth order. *E. Bompiani* (Rome).

Dalla Volta, Vittorio. Sull'isometria di calotte superficiali. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **8**, 211–227 (1949).

Here are given details of the proofs of theorems already stated in a previous note by the author [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 381–384 (1948); these Rev. **10**, 570]. *J. L. Vanderslice.*

Geĭdel'man, R. M. On congruences of circles having a single family of canal surfaces. Doklady Akad. Nauk SSSR (N.S.) **70**, 369–372 (1950). (Russian)

The paper is concerned with congruences of circles for which one "developable" surface is a canal surface. Using the method of exterior forms and a conformal focal frame, such a congruence is given by $\omega^3=0$, $\omega_1^3=a_1^2\omega_1^3+b_1^2\omega^3$, $\omega_3^0=a_3^0\omega_1^3+b_3^0\omega^2$, $\omega_2^0=q\omega_3^0$. The equations of the developable surfaces are given in terms of the ω's and by imposing various conditions on the focal surfaces (conditions on the a's, b's and q) a number of theorems on these surfaces are obtained. *M. S. Knebelman* (Pullman, Wash.).

Bompiani, Enrico. Geometria degli elementi cuspidali nel piano. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 185–191 (1950).

Let E_7 be a cuspidal element of the projective plane having the origin of nonhomogeneous coordinates as center: $a_6x^6+a_7y^7+a_8x^7y^7+\cdots$. The coefficients a_0, a_1, a_2 are regarded as homogeneous coordinates of E_7. It determines an invariant line L, the locus of the flex points of the cubics containing E_7. This paper studies the totality of E_7 (a) with the origin and L in common, (b) with only the origin in common, under (1) the collineation group, (2) all point transformations. The geometry of case (a, 1) is equivalent to the geometry of a projective plane with respect to the group G_3 of collineations leaving invariant two lines and a point on one of them. In the case (a, 2) the group G_3 becomes the group G_4 leaving two lines invariant. The cases (b, 1) and (b, 2) lead respectively to two special groups G_4, G_{10} of birational transformations of a V_2^3 (in S_3) into itself. *J. L. Vanderslice* (College Park, Md.).

Segre, Beniamino. Alcune proprietà caratteristiche delle varietà a curvatura costante. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 393–397 (1949).

This is the first of four notes [see the following reviews] all under the same title. It includes a summary of the results of all four. In an earlier paper [Boll. Un. Mat. Ital. (3) **4**, 16–22 (1949); these Rev. **10**, 736] the author proved that only surfaces of constant curvature can be mapped on the Euclidean plane with geodesics corresponding to circles (or straight lines). In this first note he proves that any Riemann space of n dimensions and of constant curvature can be so mapped on Euclidean n-space. This is accomplished by following the known mapping of geodesics into straight lines of a projective S_n' by a mapping of straight lines of S_n' on circles of a Euclidean S_n by a transformation C of the classical n-dimensional conformal group. *J. L. Vanderslice* (College Park, Md.).

Segre, Beniamino. Alcune proprietà caratteristiche delle varietà a curvatura costante. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 547–550 (1949).

[See the preceding review.] The correspondences C of part I are here characterized as those which transform lines and planes of S_n' into circles and 2-spheres of S_n. Then the author looks for line to circle correspondences not of type C and finds that such a transformation if any is a certain Cremona transformation ($n \geqq 4$) sending planes into Veronese surfaces. Such a transformation is actually set up for $n=4$ but found to be nonexistent for $n=5$. Higher values of n are not discussed. *J. L. Vanderslice.*

Segre, Beniamino. Alcune proprietà caratteristiche delle varietà a curvatura costante. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 661–667 (1949).

[See the two preceding reviews.] By a rather elaborate analysis the converse of the theorem of part I is proved for $n=3$. Namely: any Riemann 3-space admitting a geodesic to circle mapping on Euclidean 3-space is of constant curvature. *J. L. Vanderslice* (College Park, Md.).

Segre, Beniamino. Alcune proprietà caratteristiche delle varietà a curvatura costante. IV. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 12–15 (1949).

[See the three preceding reviews.] In this final note it is established that any Riemann n-space V_n, conformal to a Euclidean n-space in such a manner that geodesics correspond to lines of constant first curvature, is necessarily of constant curvature. It follows that these lines of constant curvature are circles and that the $(n-1)$-spheres of V_n correspond to hyperspheres of the Euclidean space. Five distinct canonical forms for the metric of V_n result. *J. L. Vanderslice* (College Park, Md.).

Korovin, V. I. Transformation of a line complex in a projective space with preservation of its invariant form. Doklady Akad. Nauk SSSR (N.S.) **70**, 753–755 (1950). (Russian)

A line complex is described by the edge $\gamma_{12}=[A_1A_2]$ of the fundamental tetrahedron $A_1A_2A_3A_4$. An infinitesimal displacement of this tetrahedron is given by $dA_i=\omega_i^k A_k$, where the ω's are Pfaffian forms in terms of which one obtains relative invariant forms $\varphi=\omega_1^3\omega_2^4+(\omega_2^3)^2$, $f=3\psi-(\eta+4\gamma)\varphi$ where $\psi=\omega_1\omega_1^3+\omega_2\omega_2^3+\omega_3\omega_2^4$. The quantity $I_1=f^2/\varphi$ is an absolute invariant. The main theorems of the paper are as follows. If a one-to-one reciprocal correspondence between the rays of two complexes γ_{12}, γ_{12}' can be established so that $I_1=I_1'$, the complexes are projectively equivalent. If $I_1'=\rho I_1$ there are five types of correspondences which are enumerated in terms of the inflexional centers in the ray $[A_1A_2]$. *M. S. Knebelman* (Pullman, Wash.).

Šapiro, Ya. L. Included displacements in the geometry of projective connections. Doklady Akad. Nauk SSSR (N.S.) **70**, 573–576 (1950). (Russian)

A system of curves given by $d^2x^i/dt^2=f^i(x, dx/dt)$ in V_n is said to be included in a linearly connected space V_{n+m} if the n given equations can be augmented by m others so that the enlarged system should represent geodesics in V_{n+m} (the author probably means paths). For the projective case the author considers, besides the projective equations of paths, two other included systems: type B defined by

$$\frac{d^2x^i}{dt^2}+\Pi^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=K^i+Z_{jkl}\frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^l}{dt}\frac{dx^i}{dt}$$

and type C defined by

$$\frac{d^2x^i}{dt^2}+\Pi^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=K^i+\theta_j^i\frac{dx^j}{dt}+Z_{jkl}\frac{dx^k}{dt}\frac{dx^l}{dt}\frac{dx^i}{dt}.$$

The investigation of these problems is accomplished by considering the displacement (1) $d\lambda^i/dt=f^i(x, dx/dt, \lambda)$, $i=1, 2, \cdots, n$ and the general parallel displacement

$$\frac{d^2\lambda^\alpha}{dt^2}+L^\alpha_{\beta\sigma}\frac{dx^\beta}{dt}\lambda^\sigma=A\lambda^\alpha,\quad \alpha=1, \cdots, n+m, \tag{2}$$

for a normalized vector λ^α in V_{n+m}, i.e., for a vector satisfying (3) $\phi(\lambda^\alpha, x^\beta)=0$, $(\partial\phi/\partial\lambda^\alpha)\lambda^\alpha\neq 0$. The displacement (1) is included if (1) is a consequence of (2) and (3). Any included displacement yields another defined by $\lambda^i=f(x, \mathring{\lambda})\mathring{\lambda}^i$ and for displacements of type C this gives $\mathring{\lambda}^i=\lambda^i/(1+\mu_j\lambda^j)$. For the case $m=1$ this includes the usual method of including projective displacements in affine V_{n+1}; in this case $\mu_j=\partial\rho(x)/\partial x^i$ with $\bar{x}^0=x^0+\rho(x)$.

M. S. Knebelman (Pullman, Wash.).

Varga, Ottó. Bemerkung zur Arbeit des Herrn A. Dinghas "Zur Metrik nichteuklidischer Räume." Math. Nachr. **2**, 386–388 (1949).

Dinghas benutzt in der im Titel angeführten Arbeit [Math. Nachr. **1**, 287–291 (1948); diese Rev. **10**, 571] einen speziellen Ausdruck für das Riemann'sche Bogen-

element eines n-dimensionalen Raumes von konstanter Krümmung, ohne über die Natur der zugrunde liegenden Koordinaten irgendwelchen Aufschluss zu geben. Verf. gibt eine geometrische Interpretation im Falle $K>0$, indem er als Modell eine Kugeloberfläche heranzieht und deren Punkte durch senkrechte Projektion auf eine umschriebene Zylinderoberfläche abbildet. *C. Y. Pauc* (Kapstadt).

Morduhaĭ-Boltovskoĭ, D. Bertrand curves in Lobačevskiĭ space. Doklady Akad. Nauk SSSR (N.S.) **69**, 729–730 (1949). (Russian)

The author extends the concept of Bertrand couples to Lobačevskiĭ space. The distance between the two curves is constant and there exists a linear relation between the hyperbolic cotangents of the radius of curvature and the radius of torsion. *H. A. Lauwerier* (Amsterdam).

Wrona, Włodzimierz. On multivectors in a V_n. II. Nederl. Akad. Wetensch., Proc. **52**, 29–36 = Indagationes Math. **11**, 61–68 (1949).

L'auteur continue une étude précédente [mêmes Proc. **51**, 1291–1301 = Indagationes Math. **10**, 435–445 (1948); ces Rev. **10**, 479] en passant en revue différents types de conditions locales pour qu'un V_n soit einsteinien, ou conforme à un espace euclidien ou soit un C_n. Il étudie en détail les m-vecteurs principaux d'une variété riemannienne V_{2m}, quand celle-ci est einsteinienne ou est un C_{2m} ou est hypersurface d'un espace euclidien. *A. Lichnerowicz* (Paris).

***Lichnerowicz, A. Eléménts de Calcul Tensoriel.** Librairie Armand Colin, Paris, 1950. 216 pp. 180 francs.

This small book contains an introduction to Riemannian geometry and to some of its physical applications. The approach is that of É. Cartan, whose method of the "repère mobile," satisfying in its naturalness, is applicable to more general types of geometry. Apart from its intrinsic merits, not the least of which is its simple and clear style, the book therefore provides a good introduction to the works of Cartan, even though limitations of space have prevented the author from giving more than a brief outline of some of the topics selected for discussion.

The first chapter deals with vector spaces in general and with Euclidean vector space in particular, and introduces the concept of covariance and contravariance. By admitting that the norm of a vector may be represented by an indefinite quadratic form, the author prepares the ground for the introduction of Riemannian spaces of indefinite metric. The second chapter, a short one, is concerned with the definition and properties of affine and Euclidean point-spaces. The third chapter gives an account of tensor algebra, a tensor being defined as an element of the product of two or more vector spaces. The laws of operation (addition, multiplication, contraction, etc.) are derived, and the chapter concludes with an introduction to the algebra of exterior forms. To this algebra the author calls attention in the preface, for he thinks its methods worthy of closer attention from physicists than they have so far received. Chapter IV deals with differentiation and with curvilinear coordinates in Euclidean space, that is, it gives an introduction to tensor calculus (as distinct from algebra) and to the ideas of tensor fields and of covariant differentiation. This is followed in chapter V, which is the last of part I of the book, by an account of Riemannian spaces, based upon the concepts of tangent and osculating Euclidean metrics. Geodesics and curvature are among the topics dealt with.

The remainder of the book (part II) is devoted to applications. Chapter VI indicates how the tensor calculus may be applied to classical dynamics. It is shown how the configuration space of a holonomic dynamical system may be made Riemannian by endowing it with a metric (the "kinematic metric"), and how the principle of least action is related to the "action metric," though the author does not use these terms. A short account is then given of configuration space-time, and, lastly, of the dynamics of continuous media. Chapter VII is devoted to special relativity, the main sections dealing respectively with (i) basic principles; (ii) the Lorentz group and Minkowski space-time; (iii) special-relativity dynamics of a particle and of continuous media; (iv) the Maxwell-Lorentz equations for an electromagnetic field. Chapter VII, eight pages long, gives a sketch of the general-relativity theory of gravitation. The book concludes with a short bibliography. *H. S. Ruse.*

Kilmister, C. W. The use of quaternions in wave-tensor calculus. Proc. Roy. Soc. London. Ser. A. **199**, 517–532 (1949).

The author sets up a one-to-one correspondence between quaternions over the complex field and vectors in a four-dimensional affine space. He then discusses the Dirac equations with this formalism. Presumably the affine space is a real space; however, the reality conditions are never explicitly introduced or discussed. The affine space is characterised by an enuple of vectors and an affine connection. Relations are given connecting covariant and contravariant vectors in the affine space by means of a special quadratic form in the components of the enuple vectors. The use of this particular form is not justified or discussed. However, its introduction is equivalent to the introduction of a metric in the affine space and therefore the reviewer cannot agree with the statement of the author that this work is independent of the metric. *A. H. Taub* (Urbana, Ill.).

***Doucet, E. Sur la décomposition d'un champ de vecteurs en une somme d'un rotationnel et d'un gradient.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 330–340.

It is well known that a vector field may be written as the sum of two vector fields, one of which is irrotational and the other of which has zero divergence (divergence-free). The author attempts to characterize the divergence-free vector field by use of an operational type of integration scheme. He calls this integration, "part integration." By use of some of the properties of this integration, he succeeds in obtaining the desired characterization. *N. Coburn.*

Racah, Giulio. On the decomposition of tensors by contraction. Rev. Modern Physics **21**, 494–496 (1949).

The author calculates the decomposition of tensors into irreducible representations of the orthogonal groups for three and for four dimensions. The proper (+) orthogonal group $O^+(4)$ (for four dimensions) is known to be homomorphic with the direct product of two unimodular groups in two dimensions. The author uses a direct calculation in the three-dimensional case and a method proceeding by successive steps in the four-dimensional case. The connection is shown with the problem of the allowed values of ordinary and isotopic spin for a given symmetry of the spatial eigenfunction of a nuclear system. *M. Pinl.*

Craig, Homer V., and Guy, William T., Jr. Jacobian extensors. Amer. J. Math. **72**, 229–246 (1950).

The authors extend the theory of extensors (tensors under the extended point transformations) to Jacobian extensors. Essentially, Jacobian extensors are extensors whose transformation law depends upon the weighted Jacobian of the coordinate transformation and the derivatives of this weighted Jacobian with respect to a curve parameter t. First the authors show that their transformation law is transitive. It is shown that sums, products and contractions of Jacobian extensors furnish Jacobian extensors. In particular, it is shown that Jacobian extensors contain the ordinary weighted tensors (densities and capacities) as special cases. In concluding the first section of the paper, the authors show how Jacobian extensors may be constructed from ordinary weighted tensors. The second part of the paper is concerned with the study of Jacobian connections and their role in the construction of intrinsic extensive derivatives of Jacobian extensors. *N. Coburn.*

Bruins, E. M. Line geometry and quantum mechanics. Nederl. Akad. Wetensch., Proc. **52**, 1135–1143 = Indagationes Math. **11**, 409–417 (1949).

The author discusses the Dirac equations in terms of the well-known Plücker-Klein correspondence between lines of a three-dimensional projective space and points of a quadric in a five-dimensional projective space. This has been done more thoroughly by Veblen [Proc. Nat. Acad. Sci. U. S. A. **19**, 503–517 (1933)]. *A. H. Taub* (Urbana, Ill.).

Duffin, R. J. On wave equation vector-matrices and their spurs. Physical Rev. (2) **77**, 683–685 (1950).

If a system of tensor or spinor equations may be written in the form $P\psi=\psi'$, where the column matrices ψ and ψ' undergo the same transformation under a change of coordinates, then the square matrix P is termed a vectrix. This paper is concerned with finding various identities satisfied by such matrices P for arbitrary values of the parameters. *A. H. Taub* (Urbana, Ill.).

NUMERICAL AND GRAPHICAL METHODS

Uhler, Horace Scudder. Table of exact values of high powers of 2. Scripta Math. **15**, 247–251 (1949).

The numbers 2^n for $n=778$, 889, 971, 1000, 2000, 2222, 3000, 3889 and 4001 are given. The latter number has 1205 decimal digits. *D. H. Lehmer* (Berkeley, Calif.).

Gonzalez del Valle, A. Electronic calculators. Revista Mat. Hisp.-Amer. (4) **9**, 97–109 (1949). (Spanish)

Shannon, Claude E. Programming a computer for playing chess. Philos. Mag. (7) **41**, 256–275 (1950).

Palm, F. W. Über die Verallgemeinerung des graphischen Verfahrens von Lill. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **157**, 79–96 (1949).

The graphical computation of the rational function

$$y=a_0(z-p_1)(z-p_2)\cdots(z-p_n)+a_1(z-p_1)(z-p_2)\cdots(z-p_{n-1})+\cdots+a_{n-1}(z-p_1)+a_n$$

by Lill [C. R. Acad. Sci. Paris **65**, 854–857 (1867)], which was generalized by Willers [Methoden der praktischen Analysis, de Gruyter, Berlin, 1928], is considered. The generalized treatment is developed by the construction of polygons, geometrical transformations, and the examination of the resulting enveloping curves. *E. Frank.*

Bodewig, E. Über das Quadratwurzelziehen aus kleinen Zahlen auf der Rechenmaschine. Z. Angew. Math. Mech. **29**, 377–379 (1949).

From a study of Gauss' methods of numerical calculation [P. Maennchen, Gauss als Zahlenrechner, Gauss' Werke, v. X_2, no. 6] a method is developed for the computation of square roots which is especially applicable to calculating machines. The method is based on the approximants of the continued fraction expansion $a+b/2a+b/2a+b/2a+\cdots$ for $x^2=(a^2+b)^2$. An example is given in which $\sqrt{6}$ is found to 50 decimal places with an eight-digit calculating machine. The method is extended to give results of much greater accuracy, and in the above illustration $\sqrt{6}$ is found correct to 255 decimal places. *E. Frank* (Chicago, Ill.).

Walker, Robert M. An analogue computer for the solution of linear simultaneous equations. Proc. I. R. E. **37**, 1467–1473 (1949).

An analogue computer is described which utilizes a manual adjustment of source voltages to provide the necessary feedback which is required to solve the equations. The machine is capable of handling a maximum of 12 simultaneous equations with real coefficients. Since punched cards are used to insert the values of the coefficients, the machine setup time is reduced considerably. No mention is made regarding the machine limitations which may exist when a system of equations is encountered which has a determinant whose value is less than 0.01. The provision for handling unstable equations has materially increased the size of the simultaneous equation solver previously described by Goldberg and Brown [J. Appl. Phys. **19**, 339–345 (1948); these Rev. **9**, 535]. *F. M. Verzuh* (Cambridge, Mass.).

Schilt, Heinz. Das Bestimmen von Linienintegralen mit Hilfe eines Integraphen. Z. Angew. Math. Physik **1**, 145–147 (1950).

For any given curve, its length S can be determined by measuring the area F of the strip bounded by the end normals and the two parallel curves at an arbitrary distance a from the curve (provided a is not greater than the least radius of curvature of the given curve). Then $S=F/2a$. If the parallel curves are taken at a distance $2a$, and the moment and moment of inertia of this wider strip are M_1 and I_1, while the corresponding quantities for the narrower strip are M and I, then the moment m and the moment of inertia i of the given curve are evaluated by $m=(8M_1-M)/12a$ and $i=(8I_1-I)/12a$. When the end normals are parallel, the simpler relation $m=M/2a$ may be used.

M. Goldberg (Washington, D. C.).

Winkler. A new type of product planimeter. Ein neuartiges Produktplanimeter. National Research Council of Canada. Division of Mechanical Engineering. Tech. Translation No. **TT-41**, ii+6 pp. (3 plates)+11 pp. (1947).

[Reproduction and translation of Wasserbau Versuchsanstalt, Kochelsee, Arch. **66/193** (1945).] An improved product integrator is described. Two movable bars A and

B traverse a rectangular table. One bar remains parallel to an edge of the table while the other bar remains parallel to the perpendicular edge. A cross-piece at the intersection of the bars is forced to trace a given curve $H(y)$ while another carriage on the bar A is forced to trace another given curve $g(y)$. Another carriage on the bar B is moved by a system of bands and pulleys so that its displacement along bar B is double the displacement of the other carriage along bar A. An integrator attached to the carriage on bar B then measures the area under the curve traced by this carriage. This area equals $2\int g(y)dH(y)$. *M. Goldberg.*

Krienes, Klaus. Ein Polarplanimeter zur Bestimmung des polaren Trägheitsmomentes. Z. Angew. Math. Mech. **30**, 62 (1950).

A polar planimeter is made to indicate a value which is a function of the area and the polar moment of inertia by continuously tilting the wheel axis so that the angle between the axis and the pole arm is double the angle between the wheel arm and the pole arm. This tilting is accomplished mechanically as in the power planimeter. Also, from two readings made about two different origins, the moment of area may be computed. *M. Goldberg.*

***Couffignal, L. Réalisation mécanique des calculs nécessités par la méthode de recherche des périodes de Mme. et M. Labrouste.** Colloques Internationaux du Centre National de la Recherche Scientifique, no. 14, Méthodes de calcul dans des problèmes de mécanique, pp. 45–48. Centre National de la Recherche Scientifique, Paris, 1949.

For the method in question see Labrouste and Labrouste, Analyse des graphiques résultant de la superposition de sinusoïdes, Presses Universitaires de France, Paris, 1943; these Rev. **8**, 408.

***Pollak, L. W., and Egan, U. N. All Term Guide for Harmonic Analysis and Synthesis Using 3 to 24; 26, 28, 30, 34, 36, 38, 42, 44, 46, 52, 60, 68, 76, 84 and 92 Equidistant Values.** Department of Industry and Commerce, Meteorological Service, Geophysical Publications, Vol. II. Stationery Office, Dublin, 1949. xx+185 pp. £ 2/2/0.

This is a continuation of Pollak and Heilfron, Harmonic Analysis and Synthesis Schedules . . . , same Publications, Vol. I, 1947; these Rev. **9**, 383.

Thornhill, C. K. The numerical method of characteristics for hyperbolic problems in three independent variables. Ministry of Supply, Armament Research Establishment, Fort Halstead, Kent, Rep. no. **29/48**, ii+13 pp. (5 plates) (1948).

The author proposes a scheme which permits one in principle to construct a numerical solution of the Cauchy problem for a quasi-linear differential equation of 2d order in 3 independent variables. The equations considered are of the form $\sum_{i,j=1}^{3} a_{ij}p_{ij}=b$, where ϕ is the unknown function and p_{ij} and p_i stand respectively for $\partial^2\phi/\partial x_i\partial x_j$ and $\partial\phi/\partial x_i$. The a_{ij} and b are assumed to be known functions of ϕ and the p_i. The method of solution consists in determining for each 3 points P_1, P_2, P_3 at which ϕ and the p_i are known a fourth point P_4 at which these quantities can be found up to terms of higher order. The point P_4 is taken as the vertex of that characteristic conoid that intersects the plane of the triangle $P_1P_2P_3$ in a curve that is either inscribed or circumscribed to the triangle. It is shown how the method applies to the determination of steady supersonic compressible flow in 3 dimensions and of unsteady compressible flow in 2 dimensions. *F. John.*

Ödman, Sven T. A. A method for solving partial differential equations with application to rectangular plates. Swedish Cement and Concrete Res. Inst. Stockholm, Bull. no. **10**, 13 pp. (1948).

The author considers the linear partial differential equation (1) $L_n(w)+F(x, y)=0$, where L_n is a linear operator of order n. He attempts an approximate solution of (1) in the form $w=u(x)v(y)$ where u and v "must satisfy the boundary conditions" (which apparently restricts the type of boundary problem to which the method may be applied). The one-variable functions u and v are then determined from the conditions

$$\int\int (L_n(uv)+F(x, y))u(x)dxdy=0,$$

$$\int\int (L_n(uv)+F(x, y))v(y)dxdy=0,$$

which lead to two ordinary differential equations for u and v, respectively, which in turn are solved by expansion into eigenfunctions. These expansions are then restrained by the further condition

$$\int\int (L_n(uv)+F(x, y))u(x)v(y)dxdy=0.$$

Since the coefficients of the equation for u depend on v and vice versa the process is iterative. The method is applied to the equation for the free vibration of a plane homogeneous plate, viz., $\Delta\Delta w-kw=0$. *H. O. Hartley* (London).

March, N. H. An improved approximate analytic solution of the Thomas-Fermi equation for atoms. Proc. Cambridge Philos. Soc. **46**, 356–357 (1950).

Eltermann, Heinz. Bestimmung einer ausgezeichneten Lösung der Integralgleichung erster Art durch schrittweise Näherung. Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig **1947**, no. 5, i+29 pp. (1947).

The author (1) gives a brief interpretation of the conditions for solvability of the integral equation of the first kind, (2) shows that the original integral equation can be replaced by the once iterated integral equation with symmetric kernel, (3) develops the process of solution with step-by-step approximations, (4) considers sundry variations and generalizations of the process, (5) draws certain conclusions from the manner in which the process of solution converges, and (6) closes with a simple example to illustrate the process. *W. E. Milne* (Corvallis, Ore.).

***Adams, Douglas P. An Index of Nomograms.** The Technology Press of Massachusetts Institute of Technology, Cambridge, Mass.; John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950. ix+174 pp. $4.00.

This work lists about 1700 references to alignment charts appearing in about 100 technical journals, chiefly American. The charts are listed alphabetically under key words and also categorically under 21 divisions ranging from "mathematics" to "food." This index should prove quite useful in locating nomograms which represent equations encountered in technical applications. *J. M. Thomas.*

***Boulanger, Georges R. Contribution à la théorie générale des abaques à plans superposés.** Robert Louis, Bruxelles; Gauthier-Villars, Paris, 1949. 117 pp. 1200 francs.

This thesis contains an extended abstract treatment of nomograms got by superposing a finite number of transparent planes upon each of which has been drawn one or more one-parameter families of curves duly labeled with the appropriate parameter values. A tangential contact consists in the tangency of two curves in the same or different planes and a punctual contact in the passage through a point of three curves in one, two or three planes. The relative position of the planes is fixed by certain fixation contacts, whereas reading the value of a dependent variable is accomplished by means of certain resolution contacts. Treated in detail are: the relation between the number of planes and the number of variables; the schematic representation of the various planes in their proper relation; modification of the order in which the planes appear, giving rise to the notion of equivalence; criteria for equivalence; grouping of contacts, identical grouping being necessary for equivalence, but sufficient only in the case of three planes; the condition that a variable can be read as an unknown; detailed study of the cases of three and four planes; nomograms in which sets of planes are grouped, that is, are regarded as forming units. In the discussion of the first topic it is shown that the nomograms are regarded as drawn on superposed transparent plane sheets. The number of variables N in an equation which can be directly represented by using n planes satisfies the inequalities $6n-4 \leqq N \leqq 9n-6$ so that certain values of N, namely, 4, 5, 6, 7, 13, are missing. *J. M. Thomas.*

Boulanger, Georges R. Vues nouvelles sur la nomographie. Rev. Gén. Sci. Pures Appl. N.S. **56**, 197–212 (1949).

This paper gives generalities based on the author's thesis [cf. the preceding review]. *J. M. Thomas.*

ASTRONOMY

Stumpff, K. Über die Wendepunkte des Azimuts der Fixsterne. Astr. Nachr. **277**, 229–232 (1949).

The article deals with the solution of a problem in spherical astronomy. The equation d^2A/dt^2, A being the azimuth of a star, is discussed. The resulting curves on the celestial sphere are given in stereographic projection.

D. Brouwer (New Haven, Conn.).

Stumpff, Karl. Neue Theorie und Methode der Ephemeridenrechnung. Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1947**, no. 1, 88 pp. (1949).

The contents of this publication have been published previously in more condensed form [Astr. Nachr. **274**, 49–68 (1943); **275**, 108–128, 203–222 (1947); these Rev. **7**, 224; **10**, 745]. *D. Brouwer* (New Haven, Conn.).

Stumpff, K. Über das Problem der Ortsbestimmung aus drei relativen Zenitdistanzen oder Azimuten. Astr. Nachr. **277**, 145–152 (1949).

The article presents a modification of a method by P. Harzer [Astr. Nachr. **192**, 107–110 (1912)] for solving the generalized problem of three altitudes, i.e., finding the latitude, the clock correction and the zero-point correction of the instrument from the observed altitudes of three stars for which the equatorial coordinates and the observed times are given. The object of this modification has been to simplify the practical application of the method. It is further shown that a similar procedure may be used for the solution of the more complicated problem of three azimuths.

D. Brouwer (New Haven, Conn.).

De Caro, E. Su un metodo dinamico per il calcolo d'orbita di un sistema binario visuale. I. Boll. Accad. Gioenia Sci. Nat. Catania (4) no. 1, 12–23 (1948).

De Caro, E., e Veca, G. Su un metodo dinamico per il calcolo d'orbita di un sistema binario visuale. II. Boll. Accad. Gioenia Sci. Nat. Catania (4) no. 1, 24–31 (1948).

De Caro, Eugenio. Su un metodo dinamico per il calcolo di orbita di un sistema binario visuale. III. Boll. Accad. Gioenia Sci. Nat. Catania (4) no. 1, 49–53 (1948).

The first part contains a method for determining the elements of an elliptic orbit of a visual double star from the relative x and y coordinates which are represented by a power series in the time. The second part gives a numerical example of the method. In the third part the application to parabolic and circular orbits is considered.

D. Brouwer (New Haven, Conn-).

Kaiser, F. Eine weitere raschfördernde Variante der Gaussschen Gleichung in der Bahnbestimmung. Astr. Nachr. **277**, 255–258 (1949).

A short numerical method is presented for the solution of the classical equation of the eighth degree that arises in the determination of orbits.

D. Brouwer (New Haven, Conn.).

Pylarinos, O. Über das Dreikörperproblem. Acta Math. **81**, 257–263 (1949).

The author deals with those particular cases of the problem of three bodies in which the plane of the three bodies passes permanently through a straight line fixed in an inertial coordinate system the origin of which coincides with the center of mass of the system. Proofs are given of three theorems. The first is an auxiliary theorem; (1) and (2) are valid independently of the law of mutual attraction. (1) If the three bodies of an isolated system remain in a straight line, this line must, relative to an inertial coordinate system with the origin in the center of mass, either remain fixed or rotate about the center of mass in a fixed plane. In the latter case the ratios of the distances between the bodies remain constant. (2) If the plane of three bodies forming an isolated system rotates, with respect to an inertial coordinate system with the origin in the center of mass of the three bodies, about an axis in this plane, then the distances of the three bodies from this axis are proportional to three quantities independent of the time. (3) If the mutual attraction between any two of three bodies is proportional to the power $-\nu+1$ of the distance between them, and the plane of the three bodies rotates about an axis in this plane, then there exists among the mutual distances a_{12}, a_{23}, a_{31} a relation $A_1 a_{23}^{\nu} + A_2 a_{31}^{\nu} + A_3 a_{12}^{\nu} = 0$ in which A_1, A_2, A_3 are quantities independent of the time, at least two of which are different from zero.

D. Brouwer (New Haven, Conn.).

Hil'mi, G. F. Dissipative motion in a system of n bodies attracting according to Newton's law. Doklady Akad. Nauk SSSR (N.S.) **71**, 847–850 (1950). (Russian)

Let r_{ij} be the distance between two particles P_i and P_j with masses m_i and m_j, respectively, and define

$$M_i=\sum_{j\neq i}m_im_j/\min_{j\neq i}\left(\frac{m_im_j}{m_i+m_j}\right),\quad \rho_i(t)=\min_{j\neq i}r_{ij},$$

$$\sigma_i(t)=\min_{j\neq i}dr_{ij}/dt.$$

Then if for each particle $\sigma_i(0)>0$ and $\sigma_i^2(0)>8M_i/\rho_i(0)$ the system is completely dissipative, i.e., $\rho_i(t)\to\infty$ as $t\to\infty$.

R. G. Langebartel (Urbana, Ill.).

Kopal, Zdeněk. Radial oscillations of compressible gas spheres. Proc. Nat. Acad. Sci. U. S. A. **36**, 72–84 (1950).

The author points out that the four cases in which the problem of radial oscillations of compressible gas spheres have been solved analytically all belong mathematically to the same type. The solutions are characterized by the possibility of expressing the eigen-amplitude of radial oscillations in terms of hypergeometric series. The author sets out to investigate whether other models exist possessing this property. He first shows that for finite values of central condensation no other models than those already investigated exist for which an amplitude of small radial oscillations are of the hypergeometric type. If there is not imposed any limit on the central condensation he arrives at a whole new family of such models. In this new family the oscillations are found to be characterized by a partly discrete and partly continuous spectrum of frequencies of free oscillations. The author points out that this represents the first known instance of compressible gas spheres which can perform small radial oscillations in any frequency. He also points out that such models, if subject to a periodic disturbance, would respond to it by resonance independent of the frequency, which might theoretically lead to a cataclysm.

G. Randers (Oslo).

Sobolev, V. V. On the diffuse reflection and transmission of light in a plane layer of a turbid medium. Doklady Akad. Nauk SSSR (N.S.) **69**, 353–356 (1949). (Russian)

In an earlier paper [same Doklady (N.S.) **61**, 803–806 (1948); these Rev. **10**, 657] the author obtained integral equations for the coefficients of brightness in a plane layer of a scattering medium in the special case of spherically symmetrical scattering. In the present note the method is generalized to the case of an arbitrary scattering law; expansions in spherical harmonics are assumed, and integral equations are obtained for their coefficients. Exact solutions are found in certain special cases for a medium of infinite optical depth. *F. Smithies* (Cambridge, England).

RELATIVITY

***Costa de Beauregard, O. La Théorie de la Relativité Restreinte.** Masson et Cie, Paris, 1949. vi+173 pp. 800 francs.

This book is mainly concerned with the special relativity formulation of classical physical theories described by tensor fields. The discussion differs from that usually given in that the author translates the differential equations satisfied by the tensors into integral conditions, the integrals being taken over arbitrary space-like hypersurfaces. The physical interpretation of the various integrals is given. Chapter I contains a brief discussion of the relativity principles of Newton and Einstein and a short review of the tensor analysis needed in subsequent work. The formalism the author uses could be simplified by the use of the numerical tensor density $\epsilon_{\mu\rho\lambda}$ to convert from a tensor to its dual. Chapter II reviews relativistic kinematics. Chapter III is concerned with electromagnetic theory for both the vacuum and a dielectric medium. Chapter IV discusses mechanics in a continuous medium and contains a discussion of the behavior of perfect fluids as well as those endowed with spin and viscosity. A fluid with spin is defined as one whose stress energy tensor is not symmetric. The author does not consider the fundamental problem of the consistency of this theory with the principle of relativity. Thus he never inquires as to whether the signal propagation in such media is always less than the velocity of light in vacuo. Chapter V is concerned with three subjects: (A) hydrodynamics as discussed by Eisenhart-Synge-Lichnerowicz, (B) action principles for point particles and (C) a review of de Broglie's wave mechanics. *A. H. Taub* (Urbana, Ill.).

Causse, M. Les éléments de la relativité cinématique. Ann. Physique (12) **4**, 760–805 (1949).

A relativity theory of dynamics is based upon the same fundamental principles as Milne's kinematical theory [Kinematic Relativity, Oxford, 1948; these Rev. **10**, 578]. Milne's procedure is followed closely, except that a particle is now assumed to have constant inertial mass instead of a mass proportional to $\xi^{\frac{1}{2}}$ $(=(1-v^2/c^2)^{-\frac{1}{2}})$ as in Milne's theory. One consequence of this assumption is that the energy of a particle is given by the formula $mc^2\log\xi^{\frac{1}{2}}$ instead of $mc^2\xi^{\frac{1}{2}}$. Another is that the force 4-vector $(\mathbf{F}, F_4)$ is orthogonal to the velocity 4-vector $(\mathbf{V}/\gamma^{\frac{1}{2}}, c/\gamma^{\frac{1}{2}})$; this leads to formulae

$$\mathbf{F}=k\gamma^{-\frac{1}{2}}(\mathbf{E}+c^{-1}\mathbf{V}\wedge\mathbf{H}),\quad F_4=c^{-1}\gamma^{-\frac{1}{2}}k\mathbf{E}\cdot\mathbf{V},$$

where $(\mathbf{E}, \mathbf{H})$ is a 6-vector formally identified with that of an electromagnetic field. The theory thus suggests the hypothesis that all forces are electromagnetic in character. Some of the consequences of this new hypothesis are examined, particularly in relation to the interaction between two bodies. Some of the results are interpreted as explaining, among other phenomena, the magnetic field of a rotating massive body. The author considers that his results confirm Milne's original ideas. *A. G. Walker* (Sheffield).

Ives, Herbert E. The measurement of the velocity of light by signals sent in one direction. J. Opt. Soc. Amer. **38**, 879–884 (1948).

Ives, Herbert E. Lorentz-type transformations as derived from performable rod and clock operations. J. Opt. Soc. Amer. **39**, 757–761 (1949).

It is postulated that there is a medium in which distances, times, velocities, etc., can be measured absolutely, that light has constant velocity c in this medium, and that rods and clocks moving relative to the medium experience the usual Fitzgerald-Larmor-Lorentz contractions. Distinction is made between "velocity" relative to the medium, and "rod-clock-quotient" as derived from measurements made by rods and clocks in motion. It is shown how the value of c can be determined from observations of one-way light signals made on a platform moving with uniform (but unknown) velocity. Methods outlined in the first paper are used in the second

paper to derive transformations relating distance and time measurements made by two observers in uniform relative motion. The transformations involve the self-measured velocity of a transported "setting-clock," and approximate the Lorentz transformations when this velocity is small.

A. G. Walker (Sheffield).

Wigner, E. P. Some remarks on the infinite de Sitter space. Proc. Nat. Acad. Sci. U. S. A. 36, 184–188 (1950).

The author defines a de Sitter space as an isotropic homogeneous (3+1)-space. Then, as is well known, there exist, besides the case of a Minkowski space, the finite (closed) de Sitter universe with constant positive curvature of 3-space and the infinite (open) de Sitter universe with constant negative curvature of 3-space. In the infinite de Sitter universe some time-like geodesics are closed curves of finite length. Thus, in classical physics, this universe is periodic in time. However, the covering space of this universe has infinite extension in time. The author points out the fact that in quantum mechanics the covering space may be taken to correspond to physical space and that in this case the periodicity of the universe is no longer present.

A. Schild (Pittsburgh, Pa.).

Takasu, Tsurusaburo. Sphere-geometrical unitary field theory. Proc. Nat. Acad. Sci. U. S. A. 36, 216–218 (1950).

In this short note the author summarizes some results which he has obtained in field theory. (1) The geometries of the Kaluza-Klein and Einstein spaces are equivalent to a Laguerre geometry of an Einstein space V_4. (2) The geometry of the Hoffmann space of the first kind (second kind) is equivalent to the Lie geometry (parabolic Lie geometry) of an Einstein space V_4. These results may be of some importance for conformal field theory and for further developments (meson field). *J. Haantjes* (Leiden).

Kadosch, Marcel. Sur la théorie de Birkhoff. Revue Sci. 86, 707–710 (1948).

On the basis of Birkhoff's theory, the author obtains the equations

$$\frac{d}{dt}(m'\mathbf{v}) = m' \operatorname{grad} \Phi + m'c^{-1}\mathbf{v} \wedge (\operatorname{grad} \Phi \wedge c^{-1}\mathbf{v})$$

for the motion of a particle of mass $m' = m/(1-v^2/c^2)$ in a field, due to n static masses, of Newtonian potential $-\Phi$. Thus to the Newtonian attractive force $m' \operatorname{grad} \Phi$ is added an additional force perpendicular to the velocity, a result that may be compared with the equation

$$\frac{d}{dt}(m'\mathbf{v}) = e(\operatorname{grad} V + \mathbf{v} \wedge \mathbf{H})$$

for an electromagnetic field. Quoting J. Hély [same vol., 115–120 (1948); these Rev. 10, 157], he then finds a similar analogy between the equations of motion for a field due to moving particles and the corresponding electromagnetic equations. The gravitational equations involve a tensor-potential, and the additional forces are capable of a quantum interpretation. His conclusion is that theories that bring in only a scalar potential are more easily interpreted than those that use a potential which is partly scalar and partly tensorial. *H. S. Ruse* (Leeds).

Milne, E. A. The relativity of Galilean frames. Proc. Roy. Soc. London. Ser. A. 200, 219–234 (1950).

Kinematic relativity differs from Einstein's special relativity and general relativity in assigning a preferential role to a particular triply-infinite class of frames, there being a unique fundamental frame (observer) at any given instant at each point of space. (In special relativity there are ∞^6 fundamental frames, and in general relativity all possible frames are on the same footing.) With a particular choice of time-scale (τ-time) and associated length-scale, these fundamental frames are all relatively stationary. Frames in uniform motion with respect to these stationary frames, although they correspond to the Galilean frames of classical dynamics, are not "equivalent" to the stationary frames. To relate them to each other and to the stationary frames, the author postulates that each moving observer "picks up" τ-time (which is a public time for the stationary observers) from each stationary observer as he passes him. The present paper is devoted to exploring the consequences of this procedure. The formulae correlating the space-time coordinates assigned to the same event by different uniformly moving observers are found to differ from the Lorentz formulae, although they have closely allied properties. In particular, Einstein's velocity-addition formulae still apply, and are thus found to be more general than the Lorentz formulae themselves. The apparent contractions of time-scale and of length-scale kept by relatively moving observers are discussed de novo. A modified Minkowskian space-time interval is given by $ds^2 = (1-V^2/c^2)(d\tau^2 - \sum dx^2/c^2)$, where V is the "absolute" velocity of the observer. *G. J. Whitrow.*

Gião, Antonio. The equations of Codazzi and the relations between electromagnetism and gravitation. Physical Rev. (2) 76, 764–768 (1949).

When an N-dimensional space S is considered as a hypersurface of an $(N+1)$-dimensional space then one obtains for S both external and internal metrics which must satisfy the equations of Gauss and Codazzi. By interpreting these metrics in terms of the gravitational and electromagnetic fields the author has derived a unified field theory. The present paper deals with applications of this theory to the rotating sphere and to a spherical rotating body with no permanent magnetization. *M. Wyman.*

Narlikar, V. V., and Tiwari, Ramji. On Einstein's generalized theory of gravitation. II. Proc. Nat. Inst. Sci. India 15, 249–261 (1949).

[For part I see the same vol., 73–79 (1949); these Rev. 11, 216.] Approximate equations for the field equations of Einstein's generalized theory of gravitation are obtained by suitable assumptions and restrictions. The general solution of the approximate equations is not obtained but it is found that the solution of the equations reduces to that corresponding to a point mass if the electromagnetic field is absent and to the classical field of uniform monochromatic radiation when the gravitational field is absent.

M. Wyman (Edmonton, Alta.).

Narlikar, V. V., and Tiwari, Ramji. A particular case in Einstein's generalized theory of gravitation. Physical Rev. (2) 76, 868–869 (1949).

This paper contains a summary of results developed by the authors in the paper reviewed above. *M. Wyman.*

Narlikar, V. V., and Singh, K. P. Gravitational fields of spherical symmetry and Weyl's conformal curvature tensor. Philos. Mag. (7) 41, 152–156 (1950).

The authors show that in the general theory of relativity the Schwarzschild interior solution is the only spherically

symmetric solution of physical interest which is conformal to flat space time. *M. Wyman* (Edmonton, Alta.).

Einstein, A. The Bianchi identities in the generalized theory of gravitation. Canadian J. Math. **2**, 120–128 (1950).

In attempting to generalize the general theory of relativity, based on a symmetric tensor, to a unified field theory based on a nonsymmetric tensor, it is found that one may derive many covariant equations which can serve as field equations. The present paper shows that, from a formal point of view, the field equations of the author's latest unified field theory seem to be the most natural of the many possibilities. *M. Wyman* (Edmonton, Alta.).

Fourès-Bruhat, Yvonne. Théorème d'existence pour les équations de la gravitation einsteinienne dans le cas non analytique. C. R. Acad. Sci. Paris **230**, 618–620 (1950).

The present paper provides an existence theorem for the solution of the field equations of general relativity for empty space. *M. Wyman* (Edmonton, Alta.).

Lanczos, Cornelius. Lagrangian multiplier and Riemannian spaces. Rev. Modern Physics **21**, 497–502 (1949).

Instead of operating with the components g_{ik} of the metrical fundamental tensor the author considers the components R_{iklm} of the Riemannian curvature tensor as the metrical quantities restricted by the Bianchi identities which are the auxiliary conditions of the variational problem. This gives rise to a Lagrangian multiplier which introduces a new fundamental field tensor. The fundamental field equations of the new formulation appear as differential equations of only second order of the same general structure as the customary equations of mathematical physics although the author considers it plausible that the basic invariant of the action principle should be quadratic rather than linear in the curvature components. In earlier investigations [Physical Rev. (2) **39**, 716–736 (1932); **61**, 713–720 (1942); these Rev. **4**, 56] the author obtained field equations of the fourth order from such a quadratic action principle. This made any comparison with the classical field equations of mathematical physics difficult. The present investigation completely eliminates this difficulty. The author defines first of all the "dual" curvature tensor R^*_{iklm} by the following operation:

$$R^*_{iklm}=R^{\alpha\beta\mu\nu}\delta_{\alpha\beta ik}\delta_{\mu\nu lm},$$

where $\delta_{\alpha\beta\mu\nu}$ denotes the completely antisymmetric "determinant tensor." The Bianchi identity for this new tensor appears as a divergence equation $R^{*\sigma}_{iklm,\sigma}=0$ (the "comma" refers to covariant differentiation). These equations are the auxiliary conditions of the variation problem of the action integral $\frac{1}{2}\int(R^*_{iklm})^2d\tau$. They have to be added to the action integral in the form $-\int H^{m}_{ik,\sigma}R^{*}_{iklm}d\tau$. The Lagrangian multiplier H^m_{ik} is a tensor of third order antisymmetric in i and k. Performing the variation the author obtains:

$$(*)\qquad R^*_{iklm}=H^{m}_{ik,\,l}-H^{n}_{ik,\,m}+H^{i}_{mn,\,k}-H^{k}_{mn,\,i}$$

similar to the definition of R_{iklm} on the basis of Γ-quantities. The "cyclic identity" $R^*_{iklm}\delta^{iklm}=0$ leads to the "conservation law" $H^{*}_{\sigma,\,\sigma}=0$ of the vector $H^{*}_{i}=H^{\alpha}_{\beta\gamma}\delta_{\alpha\beta\gamma i}$. The Bianchi identity leads to the basic field equations

$$P^{m}_{ik}=\Delta H^{m}_{ik}-H_{ik,\,m}+H^{m}_{i,\,k}-H^{m}_{k,\,i}=0,\qquad \Delta=\partial^2/\partial x_\alpha^2,$$

where two independent tensors of second order H_{ik} and H^{m}_{i} are introduced on the basis of the divergence operations

$$H_{ik}=H^{\sigma}_{ik,\,\sigma},\quad H^{m}_{i}=H^{i}_{m\sigma,\,\sigma}.$$

The author thus obtains a system of 24 partial differential equations of second order for the determination of the 24 field quantities H^m_{ik}. This field quantity is a tensor in analogy to the fundamental Λ-tensor of Einstein's theory of distant parallelism. The field equations are tied together by 13 identities. The principal difference between this theory and the Einstein theory of "distant parallelism" can be characterized as follows. In Einstein's theory the Λ^m_{ik} tensor is not a primitive field quantity but derived by differentiation from the fundamental "n-Bein" ph^{ν}_{i}. In the present theory the tensor H^m_{ik} is a primitive quantity, of the nature of a "generating function." The theory is strictly adapted to the mathematical nature of a four-dimensional manifold. *M. Pinl* (Dacca).

MECHANICS

***Hamel, Georg. Theoretische Mechanik. Eine einheitliche Einführung in die gesamte Mechanik.** Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LVII. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1949. xvi+796 pp. 63 DM; bound, 66 DM.

This text on theoretical mechanics starts with Newton's laws and point mechanics and proceeds to the methods of Lagrange and Hamilton, minimal principles, and nonholonomic systems. Besides rigid bodies and systems of points, an introduction to elasticity and fluid mechanics is included. The last third of the book is devoted to problems and their solutions. Many of these constitute extensions of the text proper. Based in part on lectures given at a technical school by a mathematician, the book combines mathematical rigor with good judgment of aspects of importance to the physicist and engineer. The work should prove useful to readers in any of these three fields who desire a careful and comprehensive treatment of this basic subject. *P. Franklin.*

***Berker, Ratip. Mekanik Dersleri [Lectures on Mechanics]. Vol. 1. Vectors. Kinematics.** 2d ed. Istanbul Technical University, Publ. no. 96, 1949. vii+273 pp.

This book, the first volume of a general introductory treatment of mechanics, contains two parts. The first part is concerned with vectors and the second part with kinematics. The treatment is very thorough and detailed, although some rather important topics are omitted or relegated to the exercises. Thus, in the chapter on vector algebra, the triple scalar product $a\cdot(b\times c)$ is discussed in the greatest detail, while the triple vector product $a\times(b\times c)$ is omitted from the text and is only included among the exercises. In the part on kinematics, there is an unusually detailed discussion of cycloids, epicycloids, hypocycloids, evolutes, involutes, and other curves of this nature.

D. C. Lewis (Baltimore, Md.).

***Lanczos, Cornelius. The Variational Principles of Mechanics.** Mathematical Expositions, no. 4. University of Toronto Press, Toronto, Ont., 1949. xxv+307 pp. $5.75.

The author gives an exposition of the following topics: The Euler-Lagrange equations of the calculus of variations, d'Alembert's principle, the principle of least constraint, the Lagrangian equations of motion, principle of Hamilton, principle of least action, integration by ignoration of coordinates, Legendre's transformation, the canonical equations of motion, integral invariants, canonical transformations, the brackets of Lagrange and Poisson, infinitesimal canonical transformations, the partial differential equation of Hamilton and Jacobi, solution by separation of variables, Delaunay's treatment of separable periodic systems, the significance of all this material in the development of both the older and more recent quantum mechanics. The entire approach to the subject is historical. The treatment is spirited and interesting but nevertheless steeped in tradition.

D. C. Lewis (Baltimore, Md.).

Kasner, Edward, and De Cicco, John. Survey of the geometry of physical systems of curves. Scripta Math. **15**, 193–199 (1949).

This is primarily an expository article, summarizing results which have been published in numerous earlier notes and papers [see, for instance, J. Math. Physics **26**, 104–109 (1947); Proc. Nat. Acad. Sci. U. S. A. **33**, 246–251, 338–342 (1947); **34**, 169–172 (1948); **35**, 419–422 (1949); Math. Mag. **22**, 223–233 (1949); these Rev. **9**, 110, 162, 540; **10**, 747]. In the course of the discussion a few new results are added.

L. A. MacColl (New York, N. Y.).

Bottema, O. On Grübler's formulae for mechanisms. Appl. Sci. Research A. **2**, 162–164 (1950).

This is an elementary derivation of Grübler's formulas for the least number of degrees of freedom x of a linkage of n links. If s_i are the numbers of the couples of multiplicity q_i ($q=2$ for a plane hinge, etc.) and p is the number of degrees of freedom of the free rigid body (for plane linkages, $p=3$; for spatial linkages, $p=6$), the general formula is $x=p(n-1)-\sum q_i s_i$. *M. Goldberg* (Washington, D. C.).

Milne-Thomson, L. M. The pendulum. Quart. J. Mech. Appl. Math. **2**, 479–480 (1949).

The elliptic function solution of the simple pendulum problem is obtained from the energy relation.

P. Franklin (Cambridge, Mass.).

Pugsley, A. G. On the natural frequencies of suspension chains. Quart. J. Mech. Appl. Math. **2**, 412–418 (1949).

Semi-empirical approximate expressions for the first few natural frequencies of a uniform chain are obtained.

P. Franklin (Cambridge, Mass.).

Duncan, W. J. The characteristics of systems which are nearly in a state of neutral static stability. Coll. Aeronaut. Cranfield. Rep. no. **34**, 11 pp. (1950).

A method of successive approximations, requiring only solutions of linear systems at each stage, is derived for finding the time constant of a vibrating mechanical system such that the determinant of the stiffness coefficients is nearly zero. *P. Franklin* (Cambridge, Mass.).

Sestini, Giorgio. Sopra i moti di un sistema rigido un punto del quale è vincolato ad una linea liscia. Atti Sem. Mat. Fis. Univ. Modena **3**, 18–28 (1949).

Suppose that the motion of a rigid body about a fixed point P under the action of given forces is known. The author proposes to determine the lines along which it is possible for P to move without friction and without disturbing the known rotatory motion of the body about P.

D. C. Lewis (Baltimore, Md.).

Bilimovitch, Anton. Sur le mouvement d'un corps solide avec un corps supplémentaire mobile. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **6**, 199–213 (1939).

The differential equations are set up governing the motion of a system consisting of two rigid bodies A and B subject to constraints of such a type that the relative motion of B is a given function of the motion of A. *D. C. Lewis.*

Venturelli, Lucia. Sulla riducibilità alle quadrature del moto spontaneo di un sistema rigido un punto del quale percorre una retta fissa. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **99**, 277–289 (1940).

The problem is that of a rigid body constrained to move so that one of its points (not necessarily the center of gravity) always remains on a fixed (smooth) straight line. Conditions are obtained on the structure of the body and the distribution of applied forces that such a system is reducible to quadratures. *D. C. Lewis* (Baltimore, Md.).

Manacorda, Tristano. Sul moto di un solido asimmetrico attorno ad un punto fisso. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 711–714 (1949).

Conditions are obtained in order that a rigid body with center of gravity G and with fixed point O should move in such a way that the component of the moment of momentum in the direction OG should vanish when the moment of applied forces also has a vanishing component in the direction OG. *D. C. Lewis* (Baltimore, Md.).

Sänger, Raymund. Einfluss von böenartigem Wind auf die Geschossbahn. Helvetica Phys. Acta **23**, 143–158 (1950).

Let sudden squalls of wind be superimposed upon a general uniform motion of the air. The velocity of the squalls is assumed to be independent of the altitude z. Under this assumption the author confines himself to the consideration of the dispersion of the points of impact on the horizontal (x, y)-plane through the initial point of the trajectory. The mathematical expressions of the two horizontal and mutually perpendicular velocity components $w_x(x, t)$ and $w_y(y, t)$ of the superimposed wave motion, called the squally range wind and the squally cross wind, are both assumed to be of the form

$$w_x(x, t)=w_0+A\cos[2\pi T^{-1}(t-x/u)+\delta], \qquad u=\tfrac{1}{2}gT/\pi,$$

and similarly for $w_y(y, t)$. According to experiment the period T varies between 3 and 60 sec. and the velocity amplitude A can assume values up to 10 m/sec. The results of the paper may be summarized as follows. While the range effect of a squally range wind may remain relatively small even if the period T and the amplitude A of the wind both assume their maximum values (in most cases the range effect is almost negligible), a deflection effect due to the squally cross wind may possibly constitute the main part of the actually observed deflection. *E. Leimanis.*

Boiz, Ray E. A note on the approximate plane motion during the burning period of rocket-propelled missiles launched at small angles of yaw from aircraft. J. Aeronaut. Sci. 17, 114–120 (1950).

This paper contains a solution of the equations of motion of a rocket which is valid under the following assumptions. (1) Gravity is neglected. (2) The coefficients of the cross forces and cross couples are constant. (3) The rocket's thrust is constant and large in comparison with the drag forces. (4) The change of mass of the missile during the burning period can be neglected. (5) The angle of yaw α is small enough to permit the approximations $\sin\alpha=\alpha$, $\cos\alpha=1$. (6) The missile has no yawing angular velocity at launch. (7) The rocket is axisymmetrical both in design and performance. In view of these, the assumption that the rocket is fired from an aircraft affects the initial conditions only. The solutions obtained are of the usual well-known type, being, for example, particular cases of those obtained by the reviewer [Philos. Trans. Roy. Soc. London. Ser. A. 241, 457–585 (1949), § 5; these Rev. 10, 749]. Their value lies in the fact that they are expressed in terms of three dimensionless parameters, so that, with the aid of the series of graphs given, it should be easy to apply them to a number of missiles of different design.

One minor point, which affects the numerical value of one of the author's parameters but not his general theory, seems worth mentioning. There is very little evidence, theoretical or experimental, for assuming that the coefficients of the cross-spin force and cross-velocity force are connected by the relation which he postulates, and it is only for missiles possessing lifting surfaces of relatively large area that this relation is likely to hold approximately. [See footnote on p. 480 of the paper cited, or J. B. Rosser, R. R. Newton and G. L. Gross, Mathematical Theory of Rocket Flight, McGraw-Hill, New York, 1947, pp. 29–32; these Rev. 9, 108.] *R. A. Rankin.*

Agostinelli, Cataldo. Le equazioni canoniche del moto di un sistema anolonomo come sistema associato a un determinato pfaffiano. Boll. Un. Mat. Ital. (3) 4, 345–348 (1949).

It is shown that the differential equations of motion of a nonholonomic system, reduced to the canonical form, coincide with the system of equations associated with a certain Pfaffian. *E. Leimanis* (Vancouver, B. C.).

Agostinelli, Cataldo. Sull'applicabilità del metodo di Jacobi della meccanica analitica ai sistemi anolonomi. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7, 93–95 (1949).

The equations of a nonholonomic system are written in the following "Hamiltonian" form:

$$\omega_k=\partial H/\partial p_k, \qquad k=1, 2, \cdots, n-m;$$

$$\frac{dp_k}{dt}+\frac{\delta H}{\delta\xi_k}+\sum_{j,\,k=0}^{n-m}\gamma_{j,kk}\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial p_k}=0,$$

in which for convenience $\omega_0=\partial H/\partial p_0=1$ and $q_0=t$. These equations must be completed by the equations

$$\dot{q}_i=\sum_{k=0}^{n-m}\beta_{rk}\omega_k, \qquad r=1, \cdots, n,$$

which give the velocities in terms of the parameters $\omega_0, \cdots, \omega_{n-m}$. The γ's are certain known functions and $\delta/\delta\xi_k$ stands for the linear differential operator $\sum_{r=0}^{n}\beta_{rk}\partial/\partial q_i$.
D. C. Lewis (Baltimore, Md.).

Agostinelli, Cataldo. Sull'applicabilità del metodo di Jacobi della meccanica analitica ai sistemi anolonomi. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 7, 96–99 (1949).

A rather complicated condition is given for the applicability of the method of Jacobi in which [with the notation of the preceding paper] the first order partial differential equation

$$\frac{\delta V}{\delta\xi_0}+H\left(q_0, q_1, \cdots, q_n;\ \frac{\delta V}{\delta\xi_1}, \cdots, \frac{\delta V}{\delta\xi_{n-m}}\right)=0$$

plays the role of the usual Hamilton-Jacobi partial differential equation. *D. C. Lewis* (Baltimore, Md.).

Agostinelli, Cataldo. Sul moto di un corpo rigido pesante asimmetrico col baricentro appartenente all'asse di uno dei piani ciclici dell'ellissoide d'inerzia. Atti Sem. Mat. Fis. Univ. Modena 3, 248–260 (1949).

The general motion of an asymmetrical top is considered in the case when the center of gravity lies on the axis of one of the circular sections of the ellipsoid of inertia.
D. C. Lewis (Baltimore, Md.).

Castoldi, Luigi. Equazioni lagrangiane per i sistemi a vincoli anolonomi non lineari nelle velocità. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 11(80) (1947), 189–200 (1949).

The equations described in the title are of the following form:

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{q}_i}-\frac{\partial T}{\partial q_i}=Q_i+\sum_{j=1}^{m}\lambda_j\frac{\partial\varphi_j}{\partial\dot{q}_i}, \qquad i=1, \cdots, n,$$

where the m constraint equations are $\varphi_j(\dot{q}, q, t)=0$. The author also transforms the equations by introducing so-called kinetic characteristic parameters. *D. C. Lewis.*

Hydrodynamics, Aerodynamics

***Birkhoff, Garrett. Recent developments in free boundary theory.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 7–16.

This is an expository article and a preview of the author's forthcoming Taft Lectures on Hydrodynamics. The existing status of certain parts of the theories of jets, cavities, and wakes is discussed, and the limitations of these theories are carefully examined. Recent theoretical work of various authors, and recent experimental work, in some of which the author was personally involved, are described. Serious disagreements between some theoretical and some experimental results are pointed out. The author emphasizes the roles played by the density ratio across a surface of discontinuity and the cavitation number K. In particular he mentions the physical importance of flows for which $K\neq 0$. As an application of the theory of impinging jets he discusses the use of lined cavity charges as anti-tank weapons.
E. Pinney (Berkeley, Calif.).

***Birkhoff, Garrett. Recent developments in free boundary theory.** Proc. Symposia Appl. Math., Vol. I, p. 47. American Mathematical Society, New York, N. Y., 1949. $5.25.

Cf. the preceding review.

García, Godofredo. Exact cardinal canonical equations for the finite motions and the tensions in viscous fluids. Actas Acad. Ci. Lima 12, 3–30 (1949). (Spanish)

The author obtains in vector form the equation of motion of a viscous fluid referred to moving axes. Their two-dimensional interpretation is given in Cartesian and in isometric coordinates. *L. M. Milne-Thomson* (Greenwich).

Viguier, G. Quelques aperçus sur un problème de M. J. Boussinesq. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 71–76 (1950).

The author studies the one-dimensional motion of a viscous incompressible fluid in the case where the velocity gradient is so large that one can no longer use the classical linear equation. If the motion starts from rest initially over a plane in presence of a constant body force, the no-slip condition cannot be satisfied. Nevertheless, such a mathematical problem is considered and the differential equation is shown to be reducible to that of Riccati. *Y. H. Kuo.*

Lewis, J. A., and Carrier, G. F. Some remarks on the flat plate boundary layer. Quart. Appl. Math. **7**, 228–234 (1949).

The authors discuss the solutions for the flow of a viscous incompressible fluid near the leading edge of a semi-infinite flat plate without pressure gradient. The Oseen linearization is employed which approximates the equations of motion and continuity by

$$u_0\frac{\partial u_j}{\partial x_1}+\rho^{-1}\frac{\partial P}{\partial x_j}=\nu\frac{\partial^2 u_j}{\partial x_i\partial x_i},\quad \frac{\partial u_i}{\partial x_i}=0,$$

where x_i are the coordinate directions, u_i the corresponding velocity components and u_0 the uniform free stream velocity which is parallel to the plate. Defining a perturbation stream function $\psi(x, y)$ by $u_1=u_0(1+\partial\psi/\partial y)$, $u_2=-u_0\partial\psi/\partial x$, $x=u_0x_1/\nu$, $y=u_0x_2/\nu$, the differential equation to be solved is $\nabla^2(\nabla^2-\partial/\partial x)\psi=0$ with boundary conditions $\partial\psi/\partial x=\partial\psi/\partial y\to 0$ far from the plate and $\partial\psi/\partial x=0$, $\partial\psi/\partial y=1$ when $y=0$ and $x>0$. The authors discuss the problem by applying the two-dimensional Fourier transform and obtain an explicit solution for the velocity gradient at the plate $\partial^2\psi/\partial y^2|_{y=0}=0.555x^{-\frac{1}{3}}$ which is in disagreement with the result $0.332x^{-\frac{1}{2}}$ of the Blasius solution. From this the authors conclude that it would be more appropriate to use a velocity other than u_0 in the linearization of the equations of motion and suggest replacing u_0 by cu_0, where $0\leqq c\leqq 1$. This choice does not affect the solution far from the plate but gives $\partial^2\psi/\partial y^2=(c/\pi x)^{\frac{1}{2}}$ on the plate and in comparison with Blasius solution indicates that $c=0.35$. The solution of the modified Oseen equation with this value of c then seems acceptable as the approximate solution in the region intermediate between the Stokes flow and the free stream. On the basis of these considerations, the authors suggest an iteration procedure for obtaining the exact solution for the above problem as well as a solution for the plate of finite length. *F. E. Marble* (Pasadena, Calif.).

Ulrich, A. Die ebene laminare Reibungsschicht an einem Zylinder. Arch. Math. **2**, 33–41 (1949).

The velocity distribution $U(x)$ over the surface of a symmetric cylinder may be expressed in a series of odd powers, $U(x)=u_1x+u_3x^3+u_5x^5+u_7x^7+\cdots$ where the u_n are numerical coefficients which depend on the specific shape of the cylindrical cross section. The boundary layer over this figure is found by solving the boundary layer equations using the above free stream velocity distribution. The result is the well-known Blasius series for the stream function,

$$\begin{aligned}\psi(x,\eta)=(\nu/u_1)^{\frac{1}{2}}\{&u_1xf_1+4u_3x^3f_3+6x^5(u_5g_5+(u_3^2/u_1)h_5)\\&+8x^7(u_7g_7+(u_3u_5/u_1)h_7+(u_3^3/u_1^2)k_7)\\&+10x^9(u_9g_9+(u_3u_7/u_1)h_9+(u_5^2/u_1)k_9\\&+(u_3^2u_5/u_1^2)j_9+(u_3^4/u_1^3)q_9)+\cdots\},\end{aligned}$$

where f_1, f_3, etc., are functions of $\eta=y(u_1/\nu)^{\frac{1}{2}}$ and y is the distance normal to the surface of the cylinder. These coefficients are independent of the free stream velocity distribution and hence may be calculated generally. The function f_1 is defined by the nonlinear differential equation $f_1'''-(f')^2+f_1f_1''+1=0$, $f_1(0)=f_1'(0)=0$; $f_1'(\infty)=1$, while each of the remaining functions satisfies a linear differential equation whose coefficients depend upon f_1.

The values of $f_1(\eta)$ have been calculated by Howarth [Aeronaut. Res. Comm. Report no. 1632 (1935)] while later Frössling [Lunds Univ. Årsskrift (N.F.) **36**, no. 4 (1940) = Fysiogr. Sällskapets Handlingar (N.F.) **51**, no. 4 (1940); these Rev. **2**, 331] improved Howarth's calculations and extended the computations to include f_3, g_5 and h_5. In the present paper the author calculates the values of g_7, h_7, k_7, g_9, h_9, k_9, j_9, and q_9 using Adams' method. The functions f_1, f_3, g_5, h_5, as well as their first and second derivatives, are tabulated to four decimal places. The functions g_7, h_7, k_7, g_9, h_9, k_9, j_9, q_9, and their first and second derivatives are tabulated to three decimal places; all tabulations cover the range $0\leqq\eta\leqq 4.0$ in intervals of 0.1. The results allow very simple calculations of velocity or shear distributions according to Blasius series up to the terms corresponding to x^9 in the stream function. *F. E. Marble.*

Görtler, H. Reibungswiderstand einer schwach gewellten, längs angeströmten Platte. Arch. Math. **1**, 450–453 (1949).

In previous work [Z. Angew. Math. Mech. **25/27**, 233–244 (1947); **28**, 13–22 (1948); these Rev. **9**, 392, 393] the author has discussed the boundary layer on a waved wall and in the present communication he considers a simplified calculation of the same problem. In order to reduce the complication of the analysis it is assumed that (1) the amplitude ϵ of the wave is small compared with the local boundary layer thickness and (2) the wave length λ is sufficiently short so that development of the corresponding undisturbed boundary layer velocity distribution over the length of the wave is small compared with the disturbance caused by the waved wall. Let y be the distance from the median line of the waved wall measured in terms of the boundary layer thickness $\delta_0=(\nu L/U_0)^{\frac{1}{2}}$ and x the distance along the wall measured in terms of $\lambda/2\pi$, where ν is the kinematic viscosity, L the plate length and U_0 the free stream velocity. The differential equation to be solved is $\psi_y\psi_{xy}-\psi_x\psi_{yy}-\sigma\psi_{yyy}=-\epsilon\sin x-\frac{1}{2}\epsilon^2\sin 2x$ with boundary conditions $\psi_x=\psi_y=0$ for $y=0$ and $\psi_y=1+\epsilon\cos x$ for $y\to\infty$. Since the stream function is periodic in x at large distances from the leading edge, the stream function is expressed in a Fourier series

$$\psi(x,y)=\psi_0(x,y)+\tfrac{1}{2}a_0(y)+\sum_1^\infty a_n(y)\cos nx+b_n(y)\sin nx.$$

Substituting into the differential equation, remembering that ϵ is a small quantity, a_1 and b_1 are of order ϵ while a_0, a_2 and b_2 are of order ϵ^2. The second order approximation for the stream function is then

$$\begin{aligned}\psi(x,y)=\psi_0(y)+\epsilon&[f_1(y)\cos x+g_1(y)\sin x]\\&+\epsilon^2[f_0(y)+f_2(y)\cos 2x+g_2(y)\cos 2x],\end{aligned}$$

where $\psi_0(y)$ is the stream function of the corresponding undisturbed boundary layer. The ratio of the resistance of the plane wall, $\bar{W}_R(\lambda; L)$, to that of the corresponding waved wall, $\bar{W}_R(\lambda; L)$, may be found from this result and obviously depends only on the nonperiodic part of the above result:

$$\bar{W}_R(\lambda; L)/\bar{W}_R(\lambda; L) = U_0'(0)/\{U_0'(0)+\epsilon^2 f_0''(0)\}.$$

The function $f_0(y)$ is determined through solution of the ordinary differential equations for the coefficients of the Fourier series which follow from substituting in the original partial differential equation. Numerical examples are discussed which show the frictional resistance to be decreased by the waviness. *F. E. Marble* (Pasadena, Calif.).

Kawahara, Takuma. The theory on the turbulent boundary layer and its application. J. Jap. Soc. Appl. Mech. **3**, 1–5 (1950). (Japanese. English summary)

Defining the turbulent boundary layer as the domain where eddies predominate, the author determines its thickness by solving the equations which are derived from Kármán's fundamental dissipation equations of the mean eddy energy and vorticities, and compares the results with the well-known relations. *Author's summary.*

Obuhov, A. M. The structure of the temperature field in a turbulent flow. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **13**, 58–69 (1949). (Russian)

Kolmogoroff [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 301–305 (1941); these Rev. **2**, 327] defined the notion of locally isotropic turbulence and introduced the structure function of the field of velocities of a turbulent flow. In the paper under review the author introduces the structure function of the temperature field of a turbulent flow, defined as the mean value of the square of the difference of the temperature at two points M and M': $H(M, M') = \overline{[T(M')-T(M)]^2}$. In the case of local isotropy and very large Reynolds numbers the function $H(M, M')$ depends only on the distance r between the points M and M', it being understood that both points belong to some domain and the distance r is small in comparison with the scale of the turbulence. Using a method similar to that applied by Kolmogoroff [loc. cit.] in the case of the locally isotropic field of velocities of a turbulent flow the author is led to some similar results concerning the structure function of the temperature field. For instance, if the points M and M' are "not very close together" it is found that $H(r)$ is of the form (1) $H(r) = k^2\bar{N}\bar{\epsilon}^{-1/3}r^{1/3}$, where $\bar{N}$ is the mean velocity of homogenization of the temperature field, $\bar{\epsilon}$ is the mean dissipation of energy per unit of mass, and k is a numerical constant. The dependence upon $r^{1/3}$ is the same result as that obtained by Kolmogoroff [loc. cit.] and Obuhov [Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestiya Akad. Nauk SSSR] **1941**, 453–466; these Rev. **4**, 121] for the field of velocities. For "small" values of r the approximate formula (2) $H(r) \approx \frac{1}{2}\bar{N}\lambda^{-1}r^2$ holds, where λ is the thermometric conductivity.

A scale r_1 is assigned to the characteristic of the field of pulsations of the temperature field. It is defined geometrically as the value of the abscissa of the point of intersection of the two asymptotic representations (1) and (2) of $H(r)$ for "large" and "small" values of r, respectively: $r_1 = (27k^6\lambda^3/\bar{\epsilon})^{1/4}$. Since Prandtl's number $Pr = \nu/\lambda$ is of the order of unity for the atmosphere, r_1 is of the same order of magnitude as the inner scale of turbulence (size of smallest eddies), $\eta \sim (\nu^3/\bar{\epsilon})^{1/4}$, introduced by Kolmogoroff [loc. cit.].

It is to be noticed that throughout the paper the case of fully developed turbulence of dynamical origin is considered and the pulsations of the temperature are assumed to be so small (in comparison with the mean absolute value of the temperature of the medium) that they do not influence sensitively the régime of the turbulent pulsations of the velocity of the flow. Radiative heat transfer and the heating due to dissipation of mechanical energy of the turbulence are neglected. [In formula (24) of the text and in the preceding formula there are misprints.] *E. Leimanis.*

Yaglom, A. M. On the local structure of the temperature field in a turbulent fluid. Doklady Akad. Nauk SSSR (N.S.) **69**, 743–746 (1949). (Russian)

Recently Obuhov [see the preceding review] investigated by the method of similarity and dimensional considerations the local structure of the temperature field in a turbulent flow of an incompressible viscous fluid at very large Reynolds numbers. The author improves some of these results by direct consideration of the equations of hydrodynamics. Under the assumptions made by Obuhov concerning the nature of the flow, the field of velocities satisfies the usual equations of motion of a viscous incompressible fluid, while the temperature field satisfies the equation

$$(1) \qquad \partial T/\partial t + \sum_{j=1}^{3} v_j \partial T/\partial x_j = \lambda \Delta T,$$

where λ denotes the thermometric conductivity. In the case of very large Reynolds numbers the turbulent motion may be considered as locally homogeneous and locally isotropic [Kolmogoroff, C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 301–395 (1941); these Rev. **2**, 327]. Under this assumption the author deduces the relation

$$(2) \qquad D_{lTT}(r) - 2\lambda d D_{TT}(r)/dr = -\tfrac{4}{3}\bar{N}r,$$

connecting the structure functions

$$D_{lTT}(r) = \overline{[v_l(M')-v_l(M)][T(M')-T(M)]^2},$$

$$D_{TT}(r) = \overline{[T(M')-T(M)]^2},$$

and the constant (3) $\bar{N} = \frac{3}{2}\lambda D_{TT}'(0)$ introduced by Obuhov [loc. cit.]. [For a corresponding relation between the structure functions $D_u(r)$ and $D_{uu}(r)$ of the field of velocities cf. Kolmogoroff, C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 16–18 (1941); these Rev. **3**, 221]. For small, or large, values of r it follows from (2) that (4) $D_{TT}(r) \approx \frac{1}{2}\bar{N}\lambda^{-1}r^2$ or (5) $D_{lTT}(r) \approx -\frac{4}{3}\bar{N}r$, respectively.

If F denotes the dimensionless quantity

$$(6) \qquad D_{lTT}(r)/D_{TT}(r)\{D_u(r)\}^{1/2},$$

which for large Reynolds numbers and large values of r may be considered as a negative constant, it follows from (5) and (6) that (7) $D_{TT}(r) = -\frac{4}{3}F^{-1}\bar{N}r[D_u(r)]^{-1/2}$. Using the "two-thirds law" for the field of velocities:

$$D_u(r) = C\bar{\epsilon}^{2/3}r^{2/3}, \quad C = (-4/5S)^{2/3}, \quad S = D_{uu}(r)/[D_u(r)]^{3/2},$$

the author obtains from (7) the "two-thirds law" for the temperature field [given previously by Obuhov, loc. cit.]:

$$(8) \qquad D_{TT}(r) = k^2\bar{N}\bar{\epsilon}^{-1/3}r^{2/3}, \quad k^2 = -\tfrac{4}{3}F^{-1}(-5S/4)^{1/2}.$$

The dimensionless constant k^2 used by Obuhov is now expressed in terms of the constants S and F which have simple probability-theoretical meaning. Finally an interpolation curve for $D_{TT}(r)$ is given which reduces to the asymptotic formulas (4) and (8) for their corresponding ranges of validity. *E. Leimanis* (Vancouver, B. C.).

Stewartson, K. Correlated incompressible and compressible boundary layers. Proc. Roy. Soc. London. Ser. A. **200**, 84–100 (1949).

The major part of this paper treats the compressible boundary layer under the following three assumptions: (a) the boundary layer is thermally insulating, (b) the viscosity varies as the absolute temperature, and (c) the Prandtl number of the fluid is unity. Some discussion is made with condition (a) relaxed. It is shown that under the above three conditions, for each problem of a compressible boundary layer, a corresponding problem of an incompressible boundary layer can be set up by a suitable transformation of variables. Thus, any result or development of the theory for incompressible boundary layers can be immediately applied to the study of the compressible case. This completes the correlation of these two cases first discussed by Howarth [same Proc. Ser. A. **194**, 16–42 (1948); these Rev. **10**, 270]. On the basis of this correlation, the author studies the compressible boundary layer by following the usual line of development in the incompressible case. "Similar" solutions are obtained. The Kármán-Pohlhausen procedure and the Howarth procedure for the calculation of the development of the laminar boundary layer are applied and compared for the case of linear decrease of the free stream velocity. It is concluded that the later procedure is more accurate although both seem to overestimate the position of separation. *C. C. Lin* (Cambridge, Mass.).

Prim, R., and Neményi, P. On a family of rotational spatial gas flows. Quart. J. Mech. Appl. Math. **2**, 129–135 (1949).

With the generalizations of Munk and Prim [J. Appl. Phys. **19**, 957–958 (1948); these Rev. **10**, 215] in mind, the authors investigate a class of rotational gas flows, starting with the equations of L. Crocco [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **23**, 115–124 (1936); German translation, Z. Angew. Math. Mech. **17**, 1–7 (1937)]:

$$\operatorname{div}\{(1-w^2)^{1/\gamma-1}\mathbf{w}\}=0, \quad \operatorname{curl}\left\{\frac{\mathbf{w}\times\operatorname{curl}\mathbf{w}}{1-w^2}\right\}=0.$$

The case treated is that for which the three velocity components in a cylindrical coordinate system based upon a plane isometric net depend on only one isometric coordinate. It is shown that such flows exist only if the basic net consists of logarithmic spirals or their limiting cases. Several large classes of rotational spatial flow are obtained, most of which are new. Special cases are discussed. *W. D. Hayes.*

Schultz-Grunow, F. Der Carnotsche Stossverlust in nichtstationärer Gasströmung. Z. Angew. Math. Mech. **29**, 257–267 (1949).

The author considers the pressure loss which accompanies a sudden increase in the cross-section of a pipe carrying a flow of gas. The calculation is carried out for nonstationary flows of not too rapid variability. In particular, it is shown that the phenomenon has relevance in the reflection of pressure waves at the open end of a pipe. The theory is applied to the flow of air through a periodically opened valve. The results are compared with experiment, and reasonable agreement observed. *D. P. Ling.*

Imai, Isao, and Hasimoto, Hidenori. Application of the W.K.B. method to the flow of a compressible fluid. II. J. Math. Physics **28**, 205–214 (1950).

Examples are given of the method [Imai, J. Math. Physics **28**, 173–182 (1949); these Rev. **11**, 222] of calculating subsonic gas flows by approximate formulae, based on the asymptotic behaviour of fundamental solutions of the hodograph equations. Good agreement is found, for completely subsonic flows, with solutions in powers of the Mach number, where these had been carried to high enough powers.

M. J. Lighthill (Manchester).

Sokolovskiĭ, V. V. On a transformation of the equations of plane flow of a gas. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 253–256 (1949). (Russian)

The author discusses a transformation of the equations for plane irrotational flow. The subsonic and the supersonic cases are discussed separately. The transformation is similar to those previously introduced by Chaplygin and others. Both the physical plane and the hodograph plane are considered. The author indicates how some boundary conditions may be expressed in terms of the new coordinates. However, no specific problems are completely formulated. Neither does the author indicate any special advantage of the new transformations for solving any boundary value problem. *P. A. Lagerstrom* (Pasadena, Calif.).

Guderley, G. Asymptotic representations for special solutions of the hodographic equations in two-dimensional compressible flow. TPA3/TIB Translation, GDC No. **3E/244.T**, Issued by the Ministry of Supply. (Available as **PB 96095** from Office of the Publication Board, Department of Commerce, Washington 25, D. C.) 32 pp. Undated.

This paper deals with the problem of finding the asymptotic solutions of a linear second-order differential equation, arising from the transformation of two-dimensional potential flow of gases from the physical to the hodograph plane. Following Langer's method [Trans. Amer. Math. Soc. **33**, 23–64 (1931)] the differential equation is first reduced to $z''(w)+[m^2\chi(w)+k(w)]z(w)=0$, where $\chi(w)$ and $k(w)$ are known functions of the speed w and m is a real number. For large m, the dominant term of the asymptotic form of the solution is calculated for all the speed ranges, and the errors involved in each case are estimated. The treatment appears to be quite exhaustive. To demonstrate the usefulness of the approximate solutions, Chaplygin's solution for a gas jet is shown to be convergent on the boundary.

Y. H. Kuo (Ithaca, N. Y.).

* **Nonweiler, T. An empirical treatment of some problems of transonic flow.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 437–442.

Let $a_0\phi(x, y)$ be the perturbation velocity potential function of a steady, inviscid, isentropic, irrotational flow with unperturbed Cartesian velocity components u_0, ϕ and speed of sound a_0. When the partial differential equation (1) $R\phi_{xx}-2S\phi_{xy}+T\phi_{yy}=0$ for ϕ is linearized, and if $M=u_0/a_0\sim 1$, then (2) ϕ_x is approximately independent of y. When the terms of R, S, T linear in ϕ_x, ϕ_y are retained, the author assumes that (2) still holds for $M\sim 1$, i.e., (3) $|\int(\partial\phi^{-1}/\partial y)dx|\ll 1$. If the transformation $dx'=[(\gamma+1)\phi_x]^{-\frac{1}{2}}dx$ is applied to the simplified characteristic equations $dy=\pm dx'$, it is found that the sonic line is straight, subsonic flow in the (x, y)-plane corresponds to incompressible flow in the (x', y)-plane, and supersonic flow to linearized supersonic flow at $M=\sqrt{2}$. The author outlines an example for which calculations have been performed despite his observation that it violates (3) except near the sonic line.

J. H. Giese (Havre de Grace, Md.).

***Hayes, Wallace D. Reversed flow theorems in supersonic aerodynamics.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 412–424.

Assume the linearized theory of supersonic flow. A reversed flow theorem is a theorem to the effect that some property of the flow remains invariant under reversal of the direction of the basic uniform flow. This paper presents (without details of proof) a number of reversed flow theorems. For example, the drag of certain geometrical shapes (e.g., two-dimensional systems), and the lift-curve slope of certain shapes (e.g., simple plan forms), are unchanged under flow reversal. *D. Gilbarg* (Bloomington, Ind.).

***Carrière, P. The method of characteristics applied to the problems of internal ballistics.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 3, pp. 139–153.

Traditional methods of interior ballistics such as those of Charbonnier and Sugot accept (save for correction terms given by Piobert) uniform instantaneous pressure throughout the combustion chamber and isentropic expansion. The analytic methods of Riemann, Hugoniot, Hadamard and their followers, while attempting to apply more realistic methods of gas dynamics, are asserted to assume essentially instantaneous combustion and isentropy. The author applies the method of numerical integration by characteristics, assuming that a continuous distribution of mass and of energy sources exists in the field of flow, and the problem remains one-dimensional in space. The cases of rockets and of shot in cannons are discussed separately and examples of graphical-numerical solutions are presented. The gas of combustion is treated as chemically inert, without viscosity or heat conduction. *A. A. Bennett* (Providence, R. I.).

Bleakney, Walker, and Taub, A. H. Interaction of shock waves. Rev. Modern Physics **21**, 584–605 (1949).

The authors summarize and review the theoretical and experimental work done on shock wave interaction problems. In particular, they consider in detail the central problem of shock reflection of a plane wave from a plane rigid boundary. The plane of incidence angle α versus shock strength ξ is divided in two by a curve, α-sonic, below which the shock travels faster than disturbances which it may leave in its wake. The experimental and theoretical determinations of this curve agree perfectly, and within observational accuracies it represents the upper limit of incidence angles at which regular reflection is possible. Slightly above this lies a second curve α_0 which on the basis of experiment appears to mark the lower limit of Mach (Y-type) reflection. In the no-man's-land in between lies the theoretical curve, α-extreme, above which the two-shock configuration of regular reflection fails to furnish a solution.

The theory of regular reflection is given, and this is followed by a discussion of various three-shock-plus-slip-stream theories which have been advanced to explain Mach reflection. The experimental work of one of the authors with the Princeton shock tube and of other experimenters is collected to show that, while the theory of regular reflection appears to be completely adequate, the theories of Mach reflection are in disagreement (often violent) with observation. Interferometric density patterns given in an appendix show that the usual assumptions of uniform density, or of a Prandtl-Meyer expansion, between the various shocks of a Mach configuration are incorrect. They suggest, moreover, that an approximate solution due to Bargmann (for high angles of incidence only) may point in the right direction. *D. P. Ling* (Murray Hill, N. J.).

Tsien, H. S., and Finston, M. Interaction between parallel streams of subsonic and supersonic velocities. J. Aeronaut. Sci. **16**, 515–528 (1949).

This paper investigates some of the features of the interaction between a shock wave and a boundary layer, using a simplified linearized model. The boundary layer is represented by a region of inviscid subsonic flow lying between the principal supersonic stream and a solid wall. It is found that discontinuities of pressure in the supersonic flow are smoothed out by the presence of the subsonic region, the distance upstream that the pressure disturbance is propagated being proportional to the width of the subsonic stream. The results are analogous to those of L. Howarth [Proc. Cambridge Philos. Soc. **44**, 380–390 (1948); these Rev. **10**, 270] but are closer to those expected in an actual interaction problem. *W. D. Hayes* (Providence, R. I.).

Adams, Mac C. On shock waves in inhomogeneous flow. J. Aeronaut. Sci. **16**, 685–690 (1949).

If a channel flow producing a plane normal shock is subjected to a steady local disturbance, the resulting flow inhomogeneity disturbs the shape of the shock. In this paper it is shown how to calculate by perturbation techniques the shock wave shape and conditions behind the shock. It is assumed the disturbance is small and sufficiently removed from channel walls. Linear supersonic theory and the Hugoniot equations get one downstream of the shock. This solution is then matched with a linearized subsonic solution which takes proper account of vorticity. Applications are made to various particular types of disturbances. *D. P. Ling* (Murray Hill, N. J.).

Thomas, T. Y. First approximation of pressure distribution on curved profiles at supersonic speeds. Proc. Nat. Acad. Sci. U. S. A. **35**, 619–627 (1949).

The basic configuration considered here is a curved pointed profile in a uniform supersonic flow with attached bow wave. In a paper soon to be published the author shows how the pressure at the different points of the profile can be calculated as the solution of an ordinary differential equation of infinite order. In this note he considers only the first approximation to this solution, wherein the pressure is given in closed form as a function of profile inclination. He carries through a computation which allows direct comparison with experimental measurements made by Ferri. It turns out that this simple approximation gives excellent agreement with calculations based on the flow-around-a-corner technique, and agrees with experimental measurement up to points on the profile where separation or shocks dominate the situation. *D. P. Ling* (Murray Hill, N. J.).

Spurr, Robert A., and Allen, H. Julian. A theory of unstaggered airfoil cascades in compressible flow. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **888**, 14 pp. (1947).

The Glauert thin airfoil theory modified for the effects of compressibility is used to find approximately the velocity induced at a point by an infinite unstaggered cascade of identical two-dimensional airfoils. This velocity is expressed in terms of that due to a single airfoil in a free stream. Known methods are then applied to finding the pressure distribution upon an airfoil in a given cascade and to finding the shape of the airfoil in a cascade corresponding to a given pressure distribution. Throughout the paper, the approximations made assume that the chord of the airfoil is small as compared with the distance between any two successive airfoils of the cascade. *M. Marden* (Milwaukee, Wis.).

Oswatitsch, K. Die Geschwindigkeitsverteilung bei lokalen Überschallgebieten an flachen Profilen. Z. Angew. Math. Mech. 30, 17–24 (1950). (German. English, French and Russian summaries)

In order to study the theory of thin airfoils near the transonic range, the author simplifies the equations of two-dimensional steady irrotational compressible flow to the case in which the vertical component of velocity is small and the horizontal component of velocity is near the speed of sound. He obtains the nonlinear system $U_x+V_y=UU_x$, $U_y-V_x=0$, where U and V are proportional to the differences between the x and y components of velocity locally and at infinity. The functions $U=\phi_x$, $V=\phi_y$ satisfy the second equation and for the first give $\phi_{xx}+\phi_{yy}=UU_x$. The author then applies the solution of the Poisson equation in a straightforward manner to get a nonlinear integral equation for U. The integral equation is solved approximately and the theory is applied to symmetric flow about an airfoil whose section is bounded by two equal circular arcs. The function U is calculated in several different velocity ranges. Rather pronounced shocks appear in the transonic range. The author states that the results agree with experiment.

E. Pinney (Berkeley, Calif.).

Gardner, C. S., and Ludloff, H. F. Influence of acceleration on aerodynamic characteristics of thin airfoils in supersonic and transonic flight. J. Aeronaut. Sci. 17, 47–59 (1950).

The problem attacked is that of the uniform deceleration of a thin airfoil from supersonic to transonic speed, taking advantage of the fact that for sufficiently high acceleration the transonic flow about slender bodies may be linearized [C. C. Lin, E. Reissner, H. S. Tsien, J. Math. Physics 27, 220–231 (1948); these Rev. 10, 162]. In the supersonic range the potential function due to the acceleration is calculated and expanded in powers of a dimensionless parameter proportional to the acceleration. It is found that the acceleration has little effect. In the transonic range the aerodynamic characteristics are then calculated for a diamond shaped airfoil (two wedges, back to back). These are expressed in terms of elliptic integrals which are then expanded in powers of the acceleration parameter. For one value of the acceleration parameter the coefficient of lift for a flat plate and the coefficient of drag for a wedge are plotted against the Mach number M for several aspect ratios. As $M\rightarrow 1+$, these coefficients approach a finite maximum which decreases with decreasing aspect ratio. *E. Pinney.*

***Crocco, Luigi. On a kind of stress-function for the study of non isentropic two-dimensional motion of gases.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 2, pp. 315–329.

It is well known that the equations of two-dimensional steady adiabatic flow can be written in the form of conditions that certain differential expressions are exact. The author makes use of this to express the pressure, density, and velocity components in terms of the derivatives of a real function ϕ with respect to the complex variables $z=x+iy$, $\bar{z}=x-iy$; ϕ satisfies a homogeneous nonlinear partial differential equation of the third order. The conditions on ϕ at rigid boundaries and at shock waves are developed and expressions for the force and moment on a body in terms of ϕ are given. No new problems are solved, but by way of illustration the author derives the Prandtl-Meyer solution for flow in a corner, and the purely radial flow due to a compressible point source or sink.

E. Pinney (Berkeley, Calif.).

Holt, M. The numerical method of characteristics for supersonic flows with axial symmetry. Quart. J. Mech. Appl. Math. 2, 473–478 (1949).

It is shown that the usual step-by-step process in the method of characteristics for axi-symmetric flows is less accurate than the familiar process used for plane flows. This is because the determination of the characteristic variables at a new point A, when conditions are known at two nearby points 1, 2, cannot be separated from the determination of its physical coordinates, so that an additional approximation is required. The accuracy can be improved by iteration or by the two new methods suggested by the author. To use the first of these methods, one calculates a first approximation to a value at A, starting with values at 1, 2, and then repeats the calculation using half the mesh length, starting from points 1, 2, and a point between them, say 0. The author has worked out formulas that give higher-order accuracy at A in terms of the values calculated by these two processes. The second new method avoids the two-step numerical process and involves instead the evaluation of some additional quantities at 1, 2 and, from the first approximation, A. *W. R. Sears* (Ithaca, N. Y.).

Ward, G. N. The supersonic flow past a slender ducted body of revolution with an annular intake. Aeronaut. Quart. 1, 305–318 (1950).

It is assumed that the fluid that enters the intake continues to move supersonically, at least for a short distance. The solution of the linearized small-disturbance potential equation is undertaken by Heaviside's operational methods; the appropriate solutions for the outside flow are, in operational form, modified Bessel functions. For the small radii of the body surface only the leading terms are retained. Flow at zero incidence is considered first, and both pressure coefficient and drag are found. Next, the additional effects due to incidence are calculated, including the normal force, lift, moment, and additional drag. All these are for the external flow. The normal-force or lift contribution of the internal flow is computed from momentum considerations.

W. R. Sears (Ithaca, N. Y.).

Gontier, G. Gabarit pour le tracé d'écoulements plans supersoniques. Recherche Aéronautique 1949, no. 12, 3–9 (1949).

The author has devised a drawing instrument which he calls a "caragraph" to simplify and render routine the graphical solution of problems in the two-dimensional steady compressible flow of a perfect gas of adiabatic constant 1.4. The design of the caragraph depends upon the fact that the Mach lines in the plane of flow are orthogonal to the characteristics in the hodograph plane, and that these characteristics are congruent epicycloids. Given the direction and magnitude of flow at a given point P, the corresponding point P' in the hodograph plane is determined. By means of the caragraph, the characteristics through P' may be drawn rapidly and their normal directions determined, thus giving the directions of the Mach lines through P. This device is particularly adapted to problems in which the field of flow is divided by Mach lines into a finite number of regions in which the flow is uniform, or to problems in which the field of flow may be approximated in this way. Examples

of both types are given. The accuracy appears to be rather good for so simple a device. *E. Pinney.*

Havelock, T. H. The resistance of a submerged cylinder in accelerated motion. Quart. J. Mech. Appl. Math. **2**, 419–427 (1949).

The problem considered is the resistance to motion of a circular cylinder of radius a whose center is at the constant depth f below the free surface, in particular when the cylinder starts from rest and has uniform acceleration γ. The resistance is expressed as the sum of two terms; one R_1 corresponds to the wave resistance for uniform velocity, and the other R_2 may be taken as giving an effective inertia coefficient p, the variation of which during the motion is of special interest. The expressions are carried to the second order of approximation and curves are given showing the variation of R_1 and R_2 as functions of the velocity c, and for various values of the acceleration. The author writes $R_2=\pi\rho a^2\gamma p$. The coefficient p converges to the value $1-a^2/2f^2$ when $c\to 0$ and $c\to\infty$. It is of particular interest to note that in these calculations p attains its maximum before R_1 becomes appreciable, and then as c increases falls to a minimum. Thereafter as c increases further p increases steadily towards the above limiting value.

L. M. Milne-Thomson (Greenwich).

Monin, A. S. A stationary model for the distribution of winds with altitude in the case of curvilinear isobars. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **13**, 220–237 (1949). (Russian)

L'auteur étudie le mouvement stationnaire d'un fluide pesant, visqueux et incompressible entraîné par la rotation de la terre, le coefficient de viscosité étant constant. On suppose qu'à l'infini, on se donne le champ de pression représenté par des isobares du second ordre dont un des diamètres principaux est parallèle à l'axe des x. On trouve facilement que la pression, par suite de l'équation de continuité, est un polynome du second degré en x et y; les composantes horizontales du vent sont du premier degré en x et y avec des coefficients fonctions de l'altitude, et la composante verticale fonction seulement de l'altitude. En introduisant au lieu de l'altitude une variable sans dimension s, et en substituant les valeurs des composantes de la vitesse et de la pression dans les équations du mouvement et dans l'équation de continuité et en égalent les puissances de x et y, on trouve un système complet non linéaire entre 5 des 10 inconnues; une fois résolu ce système, on obtient deux autres inconnues par des quadratures; deux autres inconnues sont déterminées par un système différentiel linéaire et la dernière par une nouvelle quadrature. Pour résoudre le système non linéaire l'auteur utilise la méthode de von Kármán dans sa théorie de la couche limite: les fonctions à étudier diffèrent de leurs valeurs à l'infini seulement à l'intérieur de la couche limite (dont il s'agit de trouver l'épaisseur) et on cherche à déterminer ces fonctions sous forme de polynomes dont les coefficients, ainsi que les équations, sont vérifiés en un point ($s=0$) de la couche limite, et tels que les équations sont vérifiées en moyenne dans toute la couche $0\leqq s\leqq 1$. On obtient de cette façon la solution cherchée. L'auteur étudie ensuite quelques cas particuliers: (1) cas des isobares linéaires, (2) cas des isobares circulaires, (3) cas des isobares hyperboles équilatères, et (4) cas des isobares paraboliques. L'auteur montre que l'on peut appliquer sa méthode dans le cas ou les isobares à l'infini sont des courbes du troisième ordre mais qui diffèrent peu, dans un domaine assez étendu autour de l'origine, de courbes du second ordre. Il étudie aussi le cas ou le coefficient de viscosité varie linéairement en fonction de s. *M. Kiveliovitch* (Paris).

Berlyand, O. S. A more exact theory of the diurnal variation of the temperature of the atmosphere taking account of the variability of the coefficient of turbulent heat exchange during the course of the day. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **14**, 70–81 (1950). (Russian)

It is assumed that an equation of the form

$$\partial\tau/\partial t=(\partial/\partial z)[k\partial\tau/\partial z]$$

($z>0$) governs the temperature τ in the earth's atmosphere, the quantity k being the coefficient of turbulent heat exchange, z the altitude. The equation $\partial\tau^*/\partial t=k^*\partial^2\tau^*/\partial z^2$ ($z<0$) governs temperature τ^* below the earth's surface, the diffusivity k^* being constant. Further, it is assumed that at the surface of the earth heat is absorbed at a rate proportional to the periodic function $j(0, t)$, giving rise to an expression for heat balance there of the form

$$(*)\qquad (1-A)j(0,t)=[\mu\tau(z,t)-\lambda\partial\tau/\partial z+\lambda^*\partial\tau^*/\partial z]_{z=0}.$$

The coefficient k is first taken as $k_1(1+\epsilon\cos\omega t)$, k_1 a constant. Then the substitution $\theta=t+(\epsilon/\omega)\sin\omega t$ leads to

$$\partial\tau/\partial\theta=k_1\partial^2\tau/\partial z^2,$$

and if $\tau=\frac{1}{2}\tau_0(z)+\sum_n\tau_n e^{-in\omega\theta}$, $\tau^*=\sum_n\tau_n^* e^{-in\omega t}$, we get the ordinary differential equations

$$k_1 d^2\tau_n/dz^2+in\omega\tau_n=0,\quad k^*d^2\tau_n^*/dz^2+in\omega\tau_n^*=0$$

to solve for the complex functions τ_n, τ_n^*. The boundary conditions (*) and $\tau(0,\theta)=\tau^*(0,t)$ are applied to the resulting series solution. Calculations carried out show some change from the case $k=k_1$. Another case considered, where k_1 is linear in z from 0 to h, constant otherwise, is naturally more complicated, but a similar procedure is used, and two computations carried out for this case. *R. E. Gaskell.*

Elasticity, Plasticity

Storchi, Edoardo. Integrazione delle equazioni indefinite della statica dei sistemi continui su una superficie di rotazione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 227–231 (1950).

This paper is concerned with the general solution of Cauchy's statical equations for a continuous medium. For two-dimensional, three-dimensional, and four-dimensional spaces of zero or constant curvature the problem was solved by B. Finzi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **19**, 578–584, 620–623 (1934)]. The author solves it for the case of a surface of revolution imbedded in a three-dimensional flat space. In general the stresses are given as linear combinations of the first, second, and third partial derivatives of a single stress function. The known special cases of the plane (Airy) and the sphere (Finzi) (and surfaces applicable to them) are exceptional in that they are the only ones in which first and second derivatives of the stress function suffice. The author indicates that for a general two-dimensional surface it can be expected that the analogous result will involve fourth derivatives also.

C. Truesdell (Washington, D. C.).

Peretti, Giuseppe. Significato del tensore arbitrario che interviene nell'integrale generale delle equazioni della statica dei continui. Atti Sem. Mat. Fis. Univ. Modena 3, 77–82 (1949).

The general solution of the equilibrium equation $t^{ij}{}_{,j}=0$ for a continuous medium subject to no extraneous force was given by Finzi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 19, 578–584 (1934)] in the form $t^{ij}=\epsilon^{ihr}\epsilon^{jks}\chi_{rs,hk}$, where χ_{rs} is an arbitrary symmetric tensor determinable only up to $u_{r,s}+u_{s,r}$, where u_r is an arbitrary vector. By suitable choice of u_r this result can be reduced to either of the classical solutions of Maxwell or Morera [Love, A Treatise on the Mathematical Theory of Elasticity, 4th ed., Cambridge University Press, 1927, § 56]. In analogy to results which the author attributes to Trefftz and Sobrero concerning Airy's stress function, he seeks a mechanical significance for the tensor $T_{ij}=\chi_{ij}+u_{i,j}+u_{j,i}$. He shows that it is possible to choose u_i in such a way that the line integral $\int T_{ij}dX^j$ taken around a closed circuit equals the moment of the stresses acting upon the inclosed area taken with respect to any origin on a certain line parallel to the resultant force of the stresses acting upon the area.

C. Truesdell (Washington, D. C.).

Gol'denblat, I. I. On a problem of the mechanics of finite deformations of continuous media. Doklady Akad. Nauk SSSR (N.S.) 70, 973–976 (1950). (Russian)

In an isotropically elastic body the stress-strain relations are of the form $t=\psi_0 I+\psi_1 e+\psi_2 e^2$, where t is the ordinary stress matrix, e is any Eulerian strain matrix, and the coefficients ψ_i are functions of the invariants of e. In order for such a relation to be derivable from an elastic potential, the ψ_i must satisfy certain conditions of integrability. For a particular choice of e the author derives these conditions. He then considers the case when the ψ_i and the strain invariants are known functions of two parameters and when an equation of state connecting these six quantities and the temperature is known. For this case he uses his conditions of integrability to obtain the coefficients ψ_i as functions of the invariants. He remarks that his method is general, but the labor of extending it to a situation in which more than two parameters are required would be forbidding. As an example he derives in this way the simple nonlinear stress-strain relationships which Murnaghan [von Kármán Anniversary Volume, pp. 121–136, Pasadena, Calif., 1941; these Rev. 3, 28] obtained by taking the classical quadratic expression for the strain as exact rather than approximate in a particular strain measure. *C. Truesdell.*

Swainger, K. H. Non-coaxiality of principal normal stresses and the 'strain' ellipsoid in the classical theory on infinitesimal deformation. Nature 165, 159–160 (1950).

Apparently the author's intention is to point out that while in an isotropically elastic body the principal axes of stress coincide with the Eulerian strain axes, these in turn do not coincide with the Lagrangian strain axes, but differ by an infinitesimal rotation. This fact is a discovery of Cauchy. The author's consequent objection to the classical infinitesimal theory is ill-taken because one of the assumptions of that theory is that the local rotation is negligible. The author's claim that "just an infinitesimal 'rotation of the body as a whole' (in my sense) through angle θ leads to loss of one-to-one correspondence between principal normal stresses and 'strains' defined by 'spatial-displacement' gradients" is not proved and in the reviewer's opinion is false.

C. Truesdell (Washington, D. C.).

Mindlin, R. D. Compliance of elastic bodies in contact. J. Appl. Mech. 16, 259–268 (1949).

A small tangential force and a small torsional couple are applied across the elliptic contact surface of a pair of elastic bodies which have been pressed together. If there is no slip at the contact surface, considerations of symmetry and continuity lead to the conclusion that there is no change in the normal component of traction across the surface and, aside from warping of the surface, there is no relative displacement of points on the contact surface. The problem is thus reduced to a "problem of the plane" in which the tangential displacements and normal component of traction are given over part of the boundary and the three components of traction are given over the remainder. In the case of the tangential force it is observed that, when Poisson's ratio is zero, the problem is a simple one, in potential theory, which is then generalized by means of a special device. An expression for tangential compliance is found as a linear combination of complete elliptic integrals. In the case of the torsional couple, the solution is obtained by generalizing a solution by H. Neuber [Kerbspannungslehre, Springer, Berlin, 1937, pp. 85–90] pertaining to a hyperbolic groove in a twisted shaft. The torsional compliance is expressed in terms of complete elliptic integrals and, for the circular contact area, reduces to that found by E. Reissner and H. F. Sagoci [J. Appl. Phys. 15, 652–654 (1945); these Rev. 6, 84]. *E. Reissner* (Cambridge, Mass.).

Timpe, A. Spannungsfunktionen für die von Kugel- und Kegelflächen begrenzten Körper und Kuppelproblem. Z. Angew. Math. Mech. 30, 50–61 (1950). (German. English, French and Russian summaries)

The author gives a number of solutions in terms of spherical coordinates for the axi-symmetrical deformations of bodies bounded by spherical and conical surfaces. The solutions are obtained on the basis of the biharmonic equation discussed in an earlier paper by the author [Z. Angew. Math. Mech. 28, 161–166 (1948); these Rev. 10, 80].

E. Reissner (Cambridge, Mass.).

Miyamoto, Hiroshi. Stress concentration around a heterogeneous insertion of ellipsoid in an infinite elastic body. I. J. Jap. Soc. Appl. Mech. 3, 15–19 (1950).

Spheroidal coordinates are used to study the stress concentration around a heterogeneous spheroidal insertion in an infinite elastic body when it is under (i) uniform axial tension, (ii) plane hydrostatic pressure in a plane perpendicular to the axis of revolution. Graphs are drawn to show how this stress concentration varies with the axial tension or the hydrostatic pressure. *B. R. Seth* (Ames, Iowa).

Finkel'štein, B. N., and Fastov, N. S. On the theory of relaxation phenomena in solid media. Doklady Akad. Nauk SSSR (N.S.) 71, 875–878 (1950). (Russian)

The strain energy of an elastic body is expressed as a linear combination of the quadratic invariants of the strain tensor and of a relaxation tensor, which is taken as a measure of the deviation from the state of static equilibrium. From this a generalized Hooke's law is derived in which the stress is defined as a function of both strain and strain rate. The number of fundamental elastic constants required to characterize a homogeneous isotropic elastic body is now increased to four to include, in addition to the Lamé constants, two new viscosity constants. The results are applied to the case of a plane transverse elastic wave in an unbounded medium. *H. I. Ansoff* (Santa Monica, Calif.).

Hemp, W. S. On the application of oblique coordinates to problems of plane elasticity and swept-back wing structures. Coll. Aeronaut. Cranfield. Rep. no. 31, 37 pp. (4 plates) (1950).

The object of this report is two-fold. On the mathematical side, it seeks to illustrate the use of oblique coordinates in applications to elasticity and structure theory. On the practical side it seeks to provide methods by which designers can solve problems of stress distribution and deflection for the case of swept-back wing structures, whose ribs lie parallel to the direction of flight. The report is divided into three parts. In part I the mathematical basis is developed. Formulae are derived which express the fundamental concepts and relations of geometry, kinetics, statics and plane elasticity in terms of vector components in oblique coordinates. In part II, the results obtained in part I are applied to a swept-back box of uniform, symmetrical rectangular section. In part III the main results of part II are generalized to cover the case of a more representative wing structure. This represents an extension of the usual engineer's theory of bending and torsion to cover the case of swept-back wings with ribs parallel to the flight direction. *E. Reissner.*

Kindem, Sverre E. Lateral buckling of beams. I. Thin-walled open sections with one axis of symmetry. Norske Vid. Selsk. Forh., Trondheim 21, no. 1, 1–4 (1948).

Kindem, Sverre E. Lateral buckling of beams. II. Norske Vid. Selsk. Forh., Trondheim 21, no. 2, 5–8 (1948).

Kindem, Sverre E. Lateral buckling of beams. III. Approximate formulae for the critical load. Norske Vid. Selsk. Forh., Trondheim 21, no. 3, 9–12 (1948).

Pozzati, Piero. La lastra rettangolare con due lati opposti semplicemente appoggiati e gli altri due sostenuti da travi elastiche e impediti o liberi di ruotare. Boll. Un. Mat. Ital. (3) 4, 374–381 (1949).

Woinowsky-Krieger, S. The effect of an axial force on the vibration of hinged bars. J. Appl. Mech. 17, 35–36 (1950).

The nonlinear problem of the free vibrations of a bar with hinged ends is reduced to a nonlinear ordinary differential equation which is solved with the aid of elliptic functions. A table gives numerical values of frequencies versus amplitude of vibration. The procedure is analogous to one employed by Kirchhoff for the corresponding problem of the vibrating string. *E. Reissner.*

Schunck, T. E. Die quadratische Platte bei Schubbelastung oberhalb der Beulgrenze. Ing.-Arch. 17, 119–128 (1949).

The author writes the nonlinear differential equations for finite deflections of thin plates in the form of three simultaneous equations for three components of displacement, as was first done by Kirchhoff. He then approximates this system of equations by a system of finite difference equations. The particular problem which is considered is that of a square plate with clamped edges acted upon by shearing stresses in the plane of the plate. The author determines the state of stress and deformation of the plate when the applied shearing stress is greater than the corresponding buckling stress. The nonlinear system of finite difference equations is solved approximately by iteration.

E. Reissner (Cambridge, Mass.).

***Plantema, F. J. Some investigations on the Euler instability of flat sandwich plates with simply-supported edges.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 200–213.

A review of published and unpublished results obtained in Holland. *E. Reissner* (Cambridge, Mass.).

***Taylor, J. Lockwood. Strength of sandwich panels.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 187–199.

A review of some English work on the subject, without reference to related work done elsewhere. *E. Reissner.*

Chien, Wei Zang. Derivation of the equations of equilibrium of an elastic shell from the general theory of elasticity. Sci. Rep. Nat. Tsing Hua Univ. Ser. A. 5, 240–251 (1948).

The macroscopic equations of equilibrium of an elastic shell are derived from the three-dimensional theory of elasticity by suitable integrations across the thickness of the shell. Tensor notation is employed. *E. Reissner.*

Zerna, W. Zur Membrantheorie der allgemeinen Rotationsschalen. Ing.-Arch. 17, 223–232 (1949).

The author reduces the problem in question to a form which in essence is similar to one given by Nemény i (as is acknowledged in a note added in proof). [See P. Neményi and C. Truesdell, Proc. Nat. Acad. Sci. U. S. A. 29, 159–162 (1943); or E. Reissner, J. Math. Physics 26, 290–293 (1948); these Rev. 5, 84; 9, 547.] He then proposes a method of numerical solution where analytical integration is impractical and carries out the solution for an example of this kind.

E. Reissner (Cambridge, Mass.).

Sonntag, G. Beanspruchung der allgemeinen, geschlossenen sowie auch offenen Kegelschale durch Belastung ihrer Spitze. Z. Angew. Math. Mech. 29, 178–185 (1949). (German. Russian summary)

The author determines the membrane stresses in conical shells acted upon by concentrated forces and moments at the apex of the cone. The problem is shown to be statically indeterminate and Castigliano's theorem of least work is used besides the equilibrium conditions for elements of the shell. *E. Reissner* (Cambridge, Mass.).

Martin, F. Die Membran-Kugelschale unter Einzellasten. Ing.-Arch. 17, 167–186 (1949).

The author reduces the problem in question to a pair of Cauchy-Riemann equations. Introducing an appropriate complex variable he obtains various singular solutions of the differential equations which are interpreted within the framework of the theory. Also considered is the problem of determining displacements which correspond to the stresses of the membrane theory. *E. Reissner.*

Drucker, D. C. Some implications of work hardening and ideal plasticity. Quart. Appl. Math. 7, 411–418 (1950).

The author first shows that the usual basic assumptions of the theory of plasticity, when coupled with a mathematical formulation of the concepts of work hardening and ideal plasticity, necessarily require the stress-strain relations to be of the Mises-Prager form. In the second part of the paper simple physical models are used to show that, whereas work hardening implies strict stability of the plastic deformation, the converse is not true. *F. B. Hildebrand.*

Drucker, D. C. Relation of experiments to mathematical theories of plasticity. J. Appl. Mech. 16, 349–357 (1949).

This paper is concerned with small elastic and plastic straining of an originally isotropic material exhibiting no time or temperature effects. Appropriate tests on thin-walled tubes subjected to tension, torsion, and internal pressure in fixed ratio are proposed, for the purpose of investigating more directly the relative significance of the octahedral-shearing-stress and the maximum-shearing-stress criteria of plastic deformation, and the range of validity of the incremental-strain theory of isotropic stress hardening. The need for a more elaborate theory, in which the anisotropy produced by the state of stress and strain is intrinsic, is indicated by the consideration of explicit experimental results. *F. B. Hildebrand* (Cambridge, Mass.).

Stockton, F. D., and Drucker, D. C. On fitting mathematical theories of plasticity to experimental results. Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Tech. Rep. **A11–44**, 21 pp. (1949).

The authors show that existing theories of plasticity cannot be made to satisfactorily fit a certain set of experimental data. Several possible explanations of this failure are advanced. *F. B. Hildebrand* (Cambridge, Mass.).

White, G. N., Jr., and Drucker, D. C. Effective stress and effective strain in relation to stress theories of plasticity. Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Tech. Rep. **A11–42**, 36 pp. (1949).

The authors propose certain definitions of effective stress and effective strain on the basis of recent stress theories of plasticity. The effective stress is essentially uniquely defined in terms of a loading function associated with the given material. On the other hand, the effective strain may be defined either under the assumption that the work dissipated depends upon the state of stress alone, and not on the path, or by taking into account the stress-strain relations. It is shown that these two alternatives lead to definitions which are in substantial agreement in certain cases, and that comparisons in other cases may be obtained experimentally. *F. B. Hildebrand* (Cambridge, Mass.).

Hodge, P. G., Jr. On torsion of plastic bars. J. Appl. Mech. 16, 399–405 (1949).

An approximate formula is derived for the determination of the warping function for a perfectly plastic bar under large angles of twist, and is applied to two examples. A geometrical process for determining this approximation in the fully plastic state is also described. In the second part of the paper it is shown that, even though a bar becomes partly plastic upon an initial loading, it may in certain cases shake down to fully elastic behavior under additional loadings and unloadings. The relevant residual stress function is found in two illustrative cases. *F. B. Hildebrand.*

Winzer, Alice, and Carrier, G. F. Discontinuities of stress in plane plastic flow. J. Appl. Mech. 16, 346–348 (1949).

A method is given for constructing the field of stress in the neighborhood of a plane discontinuity surface in problems of plane plastic flow. The magnitude of the stress discontinuity is permitted to vary along the trace of the surface in the plane of flow. It is assumed that the plastic material is governed by the Saint Venant-Mises theory. The calculation involves the determination of leading coefficients in power-series representations of the stresses, and is illustrated in a specific example. *F. B. Hildebrand.*

Batdorf, S. B. The interpretation of biaxial-tension experiments involving constant stress ratios. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2029**, 25 pp. (1950).

The slip theory of plastic flow [see Batdorf and Budianski, same Tech. Notes, no. **1871** (1949); these Rev. **10**, 648] is applied to the prediction of stress-strain relations in biaxial tension from the stress-strain curve for simple tension. Check with experiment is made for a stress-ratio of 2:1. The experimental results and the prediction of the slip theory lie between the predictions of the maximum-shear and octahedral-shear theories, and the slip theory gives the best agreement of these three. *E. H. Lee.*

Charron, Fernand. Écoulement des corpes plastiques. C. R. Acad. Sci. Paris **230**, 614–615 (1950).

A visco-plastic body is considered with a relationship between the shear stress τ, and shear strain rate $\dot{\gamma}$, of the form $\tau = f + \eta\dot{\gamma}$, where f is a friction coefficient, and η a viscosity coefficient. Steady flow down a circular cylindrical pipe produces a rigid core, the corresponding velocity distribution is determined, and an equation giving the discharge in terms of the pressure gradient. The simpler limiting form when the pressure gradient is only slightly greater than that required to start the flow is also given. *E. H. Lee* (Providence, R. I.).

Hill, Rodney. A comparative study of some variational principles in the theory of plasticity. J. Appl. Mech. **17**, 64–66 (1950).

Boundary value problems for a rigid-plastic so-called St. Venant-Mises material are considered which lead to straining throughout the body. Hill's principle of maximum rate of working is proved for the actual stress system compared with all stress systems satisfying the equilibrium equations and the yield limit which may vary in the body. A generalization of Markov's principle which determines the actual velocity distribution to give an absolute minimum for all incompressible flows satisfying the boundary conditions is also proved. It is shown that these principles give lower and upper bounds, respectively, for the rate of work over the part of the boundary on which velocities are given. Hill's principle is extended to a more general stress-strain relation based on the concept of plastic potential. Uniqueness of a solution which involves straining throughout the body is derived from both principles. *E. H. Lee.*

Greenberg, H. J. On the variational principles of plasticity. Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Tech. Rep. **A11–S4**, i+93 pp. (6 plates) (1949).

The author has provided a unified presentation of the several known variational principles in plasticity. Proofs are generally included, and relationships among the principles are discussed. Consequences of the theories of both plastic deformation and plastic flow are considered, and the principles corresponding to the respective theories are also conveniently summarized in tabular form. *F. B. Hildebrand* (Cambridge, Mass.).

Greenberg, H. J., and Prager, W. On limit design of beams and frames. Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Tech. Rep. **A18–1**, 18 pp. (6 plates) (1949).

This paper treats limit design problems which concern statically indeterminate beams or frames under the action

of given loads. The limit moments are prescribed throughout the structure, and the safety factor against collapse is required. Extremum principles which give both upper and lower bounds for the safety factor are established, and their application to the iterative calculation of safety factors is discussed and illustrated in the case of plane frameworks.
F. B. Hildebrand (Cambridge, Mass.).

Heyman, Jacques. The limit design of space frames. Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Tech. Rep. **A18–2**, 22 pp. (1949).

The author extends the application of the Greenberg-Prager maximum-minimum method [see the preceding review] to the calculation of safety factors in the limit design of space frames. The procedure is outlined specifically, and is applied to an explicit example. *F. B. Hildebrand.*

Symonds, P. S., and Prager, W. Elastic-plastic analysis of structures subjected to loads varying arbitrarily between prescribed limits. J. Appl. Mech. **17**, 9 pp. (1950).

The authors consider the analysis of structures subjected to a total load which consists of one or more component loads, whose individual maximum and minimum values only are prescribed. The explicit problem treated is that of determining whether a truss can safely carry such a set of loads, under the assumption that the materials of the component bars are ideally plastic. By defining a Euclidean "stress space" of n dimensions, in which the rectangular Cartesian coordinates represent the states of stress in the n component bars, the authors obtain a geometrical interpretation of the basic relations and accomplish a direct establishment of Bleich's theorem relevant to the existence of shake-down. Special trusses are considered in detail, and experimental tests for shake-down are discussed.
F. B. Hildebrand (Cambridge, Mass.).

Neal, B. G. Plastic collapse and shake-down theorems for structures of strain hardening material. Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Tech. Rep. **A11–40**, 36 pp. (1949).

Sufficiency conditions are established for plastic collapse and for shake-down of a truss whose members exhibit limited strain hardening. Corresponding conditions for frame structures are also discussed without proof. The results obtained generalize previous results relevant to ideally plastic materials. *F. B. Hildebrand* (Cambridge, Mass.).

Neal, B. G., and Symonds, P. S. The calculation of collapse and shake-down loads for elastic-plastic structures. Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Tech. Rep. **A11–41**, ii+55 pp. (1949).

Numerical methods for determining either the plastic collapse load or the shake-down load of a rigid frame, continuous beam, or truss, subjected to concentrated loads, are illustrated in detail in the case of a specific frame. The procedure involves a systematic reduction of a set of simultaneous linear inequalities, and is sufficiently flexible to permit efficient use of preliminary information.
F. B. Hildebrand (Cambridge, Mass.).

Swida, W. Die elastisch-plastische Biegung des krummen Stabes unter Berücksichtigung der Materialverfestigung. Ing.-Arch. **17**, 343–352 (1949).

In an earlier paper [Ing.-Arch. **16**, 357–372 (1948); these Rev. **10**, 497] the author investigated the stresses and strains in a curved bar made of a perfectly plastic material. In the present paper this analysis is extended to curved bars made of a strain-hardening material. *W. Prager.*

Lee, E. H., and Shaffer, B. W. The theory of plasticity applied to a problem of machining. Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Tech. Rep. **A11–43**, ii+57 pp. (1949).

The authors obtain analytical expressions relevant to the problem of orthogonal machining. Account is taken of the effect of the stress distribution between the work material in plastic flow and the tool face, under the assumption that the material is ideally plastic and does not exhibit work hardening. Comparisons are made with other theories, which do not explicitly consider the mode of force transmission into the work material, and the importance of this consideration is verified by reference to experimental data.
F. B. Hildebrand (Cambridge, Mass.).

Somigliana, C. Le oscillazioni sismiche e le onde di Lord Rayleigh. Rend. Sem. Mat. Fis. Milano **19** (1948), 99–117 (1949).

L'auteur reprend la théorie des ondes seismiques de Rayleigh. Il montre l'existence d'un couple d'ondes planes associées dans le même plan. Ces ondes correspondent à deux racines de l'équation de troisième ordre servant au calcul des vitesses. Ces racines et les ondes qui en résultent ne trouvent pas d'application dans la théorie de Rayleigh. L'auteur croît cependent que les seismogrammes confirment l'existence de ces ondes. *V. A. Kostitzin.*

Cambi, Enzo. Verifica sismica di massima di una torre metallica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 608–614 (1949).

L'auteur étudie les oscillations libres ou forcées d'une tour dont la base est solidement encastrée dans le sol. En remplaçant la forme réelle de la tour par une image schématisée et en admettant quelques hypothèses simples sur la distribution des masses on peut réduire le problème à une équation en dérivées partielles de deuxième ordre par rapport au temps et de quatrième ordre par rapport aux variables spatiales; dans le cas particulier considéré par l'auteur il n'y a qu'une seule variable spatiale. Cette équation se résout par rapport à cette variable en fonctions de Bessel ou autres fonctions faciles à calculer et par rapport au temps en fonctions exponentielles. L'auteur étudie le cas des oscillations forcées ayant en vue la vérification seismique de la stabilité des constructions. *V. A. Kostitzin.*

Berzon, I. S. A method of solution of the space problem of interpretation of the hodographs of Mintrop waves in the case of surfaces of discontinuity of arbitrary form. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **13**, 545–555 (1949). (Russian)

L'auteur propose une méthode de détermination d'une surface de discontinuité lorsque les vitesses $V_1(x, y, z)$ et $V_2(x, y, z)$ des ondes de Mintrop sont connues. La méthode est généralisée pour le cas d'une milieu composé des plusieurs couches. L'auteur suppose que le hodographe ainsi que la surface sur laquelle on a fait les mesures n'ont pas de points singuliers et que les vitesses V_1 et V_2 sont fonctions continues et deux fois dérivables. Dans le but de rendre les calculs plus faciles l'auteur transforme les équations différentielles du problème en équations en différences finies et utilise la méthode de Störmer de la résolution numérique des équations de ce type. *V. A. Kostitzin* (Paris).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Marx, H. Die Berechnung der Bildfehler- und Eikonal-Koeffizienten beliebiger Ordnung bei rotationssymmetrischen optischen Systemen aus Kugel- und Planflächen. Optik **4**, 97–148 (1948).

The author starts to develop a higher order aberration theory by using as variables not the entrance height at various surfaces, but the perpendiculars from the various vertices to the ray [the use of the perpendiculars from center to rays occurs in a paper by the reviewer [J. Opt. Soc. Amer. **26**, 52–62 (1936)], who applied it to third order aberration theory]. The author develops the aberrations as functions of object height and perpendicular from the entrance pupil and derives a series of recurring formulae which give the image errors of higher order as functions of those of lower order for a single surface and for rays in the meridional plane.

The second part develops the corresponding laws for off-axis rays. The reviewer is not sure that the method of the author leads to a system of independent errors, but it is evidently valuable for the approximate tracing of image errors of higher order. The same criticism applies to the study of the characteristic functions. Unfortunately, although the characteristic functions for the partial system are sums of contributions of the single surfaces, they are not functions of the same variables. The elimination of the intermediate variables is the crucial problem in the use of characteristic functions. The methods of the author seem not to give a complete solution of this problem.

The methods are to a certain extent similar to those developed by H. Buchdahl [e.g., J. Opt. Soc. Amer. **38**, 14–19 (1948); these Rev. **9**, 548]. The author, however, succeeded in applying them to a larger number of problems.

M. Herzberger (Rochester, N. Y.).

Herzberger, M., and Hoadley, H. O. The calculation of aspherical correcting surfaces. J. Opt. Soc. Amer. **36**, 334–340 (1946).

Si tratta di un metodo di approssimazioni successive, specialmente adatto per il calcolo di lamine correttrici di Schmidt. Per esempio, se la superficie da calcolare è l'ultima del sistema, si può prendere come prima approssimazione la superficie sferica che dà la voluta immagine parassiale. Si trovano i punti di intersezione di tale superficie con alcuni raggi opportunamente scelti, il cui andamento attraverso alle superficie precedenti è stato calcolato per via trigonometrica. Si determinano poi le direzioni che i raggi rifratti dovrebbero avere per congiungere i punti d'intersezione con l'immagine perfetta, e quindi l'inclinazione che la superficie dovrebbe avere in tali punti. Come seconda approssimazione si prende allora una superficie la cui freccia è rappresentata da un polinomio e che per quelle ordinate presenta quelle inclinazioni. Si determinano i nuovi punti d'intersezione e si ripete il procedimento. Il metodo è rapidamente convergente e nell'applicazione pratica si presta a evidenti semplificazioni. Gli autori lo generalizzano poi al caso in cui la superficie da correggere è una superficie interna del sistema e pertanto si deve fare in modo che i raggi emergenti da essa siano normali a una superficie d'onda di forma assegnata.

G. Toraldo di Francia (Firenze).

Keller, Joseph B., and Keller, Herbert B. Determination of reflected and transmitted fields by geometrical optics. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. EM–13. ii+15 pp. (1949).

Keller, Joseph B., and Keller, Herbert B. Determination of reflected and transmitted fields by geometrical optics. J. Opt. Soc. Amer. **40**, 48–52 (1950).

[The two papers are identical.] Let an incident electromagnetic field due to a point source impinge upon an arbitrarily curved interface between two different homogeneous isotropic media. Then [cf. R. K. Luneberg, Mathematical Theory of Optics, Brown University, Providence, R. I., 1944; these Rev. **6**, 107] geometrical optics provides approximate expressions for the magnitudes of the reflected and transmitted field components. These depend on the Jacobian $J=d\sigma/d\sigma'$, determining the transformation of cross-sectional areas $d\sigma$ and $d\sigma'$ of a tube of rays at the field point and the interface respectively. This Jacobian is evaluated in terms of geometric properties of the interface and distances independently of the choice of coordinates. In the discussion of the solution, the authors derive the equation of caustics as well as the general lens and mirror law. The calculation constitutes a check on an earlier solution in acoustics [H. Primakof and J. B. Keller, J. Acoust. Soc. Amer. **19**, 820–832 (1947); these Rev. **9**, 315].

C. Bouwkamp (Eindhoven).

Wilcock, W. L. On a paper by Vašiček concerning reflection from multilayer films. J. Opt. Soc. Amer. **39**, 889–890 (1949).

Crook aveva già notato [J. Opt. Soc. Amer. **38**, 956–964 (1948); questi Rev. **10**, 581] che la trattazione di Vašiček [ibid. **37**, 623–634 (1947); questi Rev. **9**, 105] conteneva un errore. L'autore fa notare che l'errore proviene dall'aver male applicato la formula per la riflessione da un singolo strato, ma non invalida la possibilità di ridurre successivamente la riflessione a quella di una sola superficie equivalente.

G. Toraldo di Francia (Firenze).

***Pekeris, C. L. Ray theory vs. normal mode theory in wave propagation problems.** Proc. Symposia Appl. Math. **2**, 71–75 (1950). $3.00.

A particularly simple propagation problem (point source between two infinite perfectly reflecting parallel planes) is chosen for proving that the ray and normal-mode solutions stand in a relationship of Poisson's transformation. As to the author's question whether this relationship holds in more complicated propagation problems, and his suggestion as to a possible use of the Euler-Maclaurin expansion, the reviewer would like to refer to Bremmer's monograph [Terrestrial Radio Waves, Theory of Propagation, Elsevier Publishing Co., New York, 1949, chap. IX; these Rev. **11**, 295].

C. J. Bouwkamp (Eindhoven).

Bailey, V. A. Electro-magneto-ionic optics. J. Proc. Roy. Soc. New South Wales **82** (1948), 107–113 (1949).

The problem considered is to determine the electric waves which can exist in a medium containing electrons, positive ions and molecules when it is subject to static electric and magnetic fields. It is assumed that the motion of the positive ions may be neglected. The author considers a plane wave $\exp i\omega(t-qx)$, propagated along the x-axis,

and obtains a first order perturbation correction to the static field. The condition for a finite solution of the perturbed equations is obtained, by some heavy algebraic manipulations, in the form of the existence of roots of a complex polynomial which is of the sixth degree in ω and of the eighth degree in q. For small collision frequencies the equation is separated into its real and imaginary parts, and it is found that the roots of the real part can be of two types: those in which ω and q are both real, and those in which at least one of the parameters is complex. Under certain conditions a wave-group composed of waves of the second type with frequencies in a narrow band about the frequency ω may grow in amplitude as it progresses. *M. C. Gray.*

Bailey, V. A. Plane waves in an ionized gas with static electric and magnetic fields present. Australian J. Sci. Research. Ser. A. 1, 351–359 (1948).

This is an extension of the paper reviewed above, to include the motion of the positive ions. The resulting secular equation is now a polynomial of the twelfth degree in both ω and q. A study of the roots of this equation shows that for a given real angular frequency ω most of the twelve roots q_s are real in the first approximation (neglecting collisions). When the second approximation is included the corresponding waves and wave-groups are in general attenuated as they progress. But for limited ω-bands there may be conjugate complex roots so that a narrow wave-group centered about ω will grow in amplitude. Such wave-groups occur as long as the collision frequencies are less than certain fixed numbers. Similarly for a given real wave-number q the twelve corresponding frequencies can be evaluated by successive approximations. Most of these frequencies define waves which are damped with time, but there are certain finite ranges of q-values within which the corresponding wave amplitudes increase exponentially with time. Examples of roots of this type are given, and possible applications of the theory to such diverse phenomena as low-pressure discharge tubes, the ionosphere and the solar atmosphere are indicated.

M. C. Gray (Murray Hill, N. J.).

Meixner, J. Die Kantenbedingung in der Theorie der Beugung elektromagnetischer Wellen an vollkommen leitenden ebenen Schirmen. Ann. Physik (6) 6, 2–9 (1949).

The author investigates the behaviour of electromagnetic fields in the neighbourhood of the edge of an infinitely thin, perfectly conducting screen. Without appropriate conditions at the edge, as to the character of the admissible singularities in the field, an electromagnetic diffraction problem does not have a unique solution. An illustrative example in the case of zero frequency is provided by a circular disk in a homogeneous electrostatic field, which problem is worked out in detail. Generally, it is shown that the highest admissible singularity in the electromagnetic field vectors is that corresponding to $R^{-\frac{1}{2}}$ where R is the distance to the edge [cf. C. J. Bouwkamp, Physica 12, 467–474 (1946); these Rev. 8, 363]. In particular, the component of the magnetic vector in the direction of the edge remains finite at the edge, while the component of the electric vector in the same direction vanishes as $R^{\frac{1}{2}}$. Using these results, the author proves the solution of the diffraction problem to be unique. Further, the conditions mentioned are readily expressed in terms of corresponding conditions on the Debye potentials employed in an earlier paper [J. Meixner, Z. Naturforschung 3a, 506–518 (1948); these Rev. 11, 141]. [The author wishes to correct reference 7 on page 9, which should read: A. Sommerfeld, Math. Ann. 47, 317–374 (1896).] *C. Bouwkamp* (Eindhoven).

Brillouin, L. The scattering cross section of spheres for electromagnetic waves. J. Appl. Phys. 20, 1110–1125 (1949).

According to the conventional theory of the diffraction of a plane-polarized electromagnetic wave by a perfectly reflecting metallic sphere [cf. J. A. Stratton, Electromagnetic Theory, McGraw-Hill, New York, 1941], the scattering cross-section of the sphere tends to twice its geometrical cross-section as the wavelength of the incident field tends to zero. This is in apparent disagreement with geometrical optics, according to which the scattering cross-section is equal to the geometrical cross-section in this limiting case of zero wavelength. The author points out that this is a matter of definition of the term "scattering cross-section," in that the relevant integration of Poynting's vector of the secondary field, so far as the physical concept of scattering cross-section is concerned, should be taken over all directions outside the "shadow cone." When Poynting's flux theorem is applied to the secondary field with inclusion of the shadow cone, as is usually done, the result is twice as large (in the limiting case of zero wavelength, where the shadow cone becomes a cylinder of geometrical cross-section equal to that of the sphere). This point of view is confirmed by detailed calculations. Reviewer's remark: the author is apparently unaware of the earlier discussion of the paradox of the factor 2 by H. C. van de Hulst [Optics of spherical particles, thesis, Utrecht, 1946, in particular, p. 24].

C. J. Bouwkamp (Eindhoven).

Durand, Émile. La diffraction des ondes lumineuses planes par les écrans noirs. Rev. Optique 28, 325–351 (1949).

In the Kirchhoff theory of diffraction by a black screen, monochromatic light is specified by a scalar function $\Phi e^{i\omega t}$. If Φ^0 specifies the incident light, the effect behind the screen is given by

$$(1)\quad \Phi(P)=(4\pi)^{-1}\int\int\left[\Phi^0\frac{\partial}{\partial N}(r^{-1}e^{-ikr})-r^{-1}e^{-ikr}\partial\Phi^0/\partial N\right]d\sigma,$$

where integration is over the apertures in the screen. The fact that the values taken by Φ and $\partial\Phi/\partial N$ in the apertures are not Φ^0 and $\partial\Phi^0/\partial N$ is usually regarded as a defect in the theory. But the present author thinks that "cela est tout à fait satisfaisant, car il est certain que la présence de l'écran doit modifier l'onde incidente." I cannot share this view, and prefer to regard (1) as a first approximation. If the screen lies in the plane $z=0$, (1) can be expressed in a form involving Fourier integrals, viz., $\Phi=\frac{1}{2}(\psi+\psi')$, where

$$(2)\quad \psi=\int_{-\infty}^{\infty}\int F(l,m)e^{-i(lx+my+nz)}dldm,$$

$$\psi'=\int_{-\infty}^{\infty}\int (i/n)G(l,m)e^{-i(lx+my+nz)}dldm,$$

where $n=-(k^2-l^2-m^2)^{\frac{1}{2}}$ when $z<0$. The functions F and G are given by

$$(3)\quad F(l,m)=(4\pi^2)^{-1}\int\int\Phi^0(x,y,0)e^{i(lx+my)}dxdy,$$

$$G(l,m)=(4\pi^2)^{-1}\int\int\Phi_z^0(x,y,0)e^{i(lx+my)}dxdy,$$

where integration is over the apertures in the screen. The parts of the integrals (2) which arise from the region $l^2+m^2>k^2$ give rise to evanescent waves and may be neglected at a distance from the screen large compared with the wave-length. The formulae (2) and (3) are applied to find approximate solutions of the problem of diffraction of plane waves by a half-plane, a slit, a set of parallel slits, a rectangular hole, and a circular hole. *E. T. Copson.*

Durand, Émile. Influence d'un champ quelconque sur une sphère ou sur un cylindre circulaire. C. R. Acad. Sci. Paris **230**, 188–190 (1950).

The problem is to find a solution of Laplace's equation which is equal to a constant on the conducting sphere and which has no singularity outside the sphere other than the sources of the applied field. It is solved by Kelvin's transformation. The case of a dielectric sphere and the corresponding two-dimensional problems are also considered. *E. T. Copson* (Dundee).

Fok, V. A. The field of a vertical or horizontal dipole raised above the surface of the earth. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **19**, 916–929 (1949). (Russian)

L'autore ha già elaborato in un suo libro [La diffrazione delle radioonde attorno alla superficie terrestre, Mosca, 1946; sommario in C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 310–313 (1945); questi Rev. **7**, 100] un metodo generale per sommare le serie che rappresentano il campo prodotto da un dipolo situato sulla superficie terrestre. Ora il metodo viene applicato al caso di un dipolo verticale o orizzontale, che si trovi ad una certa altezza. Non è possibile qui entrare nei dettagli del calcolo. Per il caso del dipolo verticale il vettore di Hertz U risulta nella forma

$$U=e^{ibs}\{sa\sin(s/a)\}^{\frac{1}{2}}V(x,y_1,y_2,q)$$

dove a è il raggio terrestre, $s=a\theta$ è la distanza del punto di osservazione proiettata sulla superficie terrestre, x è questa distanza opportunamente ridotta, y_1, y_2 sono rispettivamente le altezze ridotte del punto di osservazione e del dipolo e q è un parametro che dipende dalla costante dielettrica del suolo. Il fattore di attenuazione V è espresso per mezzo di un integrale contenente funzioni di Airy. Nella regione dell'ombra geometrica V può essere calcolato mediante una somma di residui, mentre per la zona illuminata si può applicare il principio della fase stazionaria, trovando naturalmente il contributo del raggio diretto e di quello riflesso. Un calcolo analogo viene effettuato per i vettori di Hertz (elettrico e magnetico) nel caso del dipolo orizzontale. Una conseguenza di questo calcolo è il graduale passaggio del campo elettrico dalla direzione orizzontale nella zona illuminata a quella verticale nell'ombra più profonda. *G. Toraldo di Francia* (Firenze).

Ott, H. Zur Reflexion von Kugelwellen. Ann. Physik (6) **4**, 432–440 (1949).

In an earlier paper [Ann. Physik (5) **41**, 443–466 (1942); these Rev. **5**, 121] the author considered the field of a vertical dipole, situated at a finite height above the interface between two media of different refractive indices, with particular reference to the existence of certain subsidiary waves, called Flankenwellen, in a limited region surrounding the angle of total reflection. The present paper extends the analysis to the case of a dipole inclined at an angle Θ to the z-axis. This increases the number of field components, but does not essentially complicate the analysis, and it is again found that the Flankenwellen exist near the angle of total reflection. The distant field of this component of the reflected wave is derived, and shown to be polarized even when the primary wave is unpolarized. *M. C. Gray.*

Brehovskih, L. M. On a new method for solving the problem of the field of a point source in a stratified inhomogeneous medium. Izvestiya Akad. Nauk SSSR. Ser. Fiz. **13**, 409–420 (1949). (Russian)

L'autore considera l'irraggiamento di un dipolo con l'asse z perpendicolare alla stratificazione del mezzo e posto all'altezza z_0. Il mezzo possiede uno strato omogeneo compreso fra $z=0$ e $z=h_0$; la propagazione dentro questo strato viene studiata scomponendo l'onda sferica emessa dal dipolo in un insieme di onde piane secondo il metodo di Weyl. Se $V_1(\alpha)$ e $V_2(\alpha)$ indicano rispettivamente i coefficienti di riflessione dello strato non omogeneo inferiore e di quello superiore in funzione dell'angolo d'incidenza, la funzione d'onda risulta in coordinate cilindriche

$$\psi=-\tfrac{1}{2}ik_0\int_\Gamma\frac{(e^{-bz}+V_1e^{bz})(e^{-b(h_0-z_0)}+V_2e^{b(h_0-z_0)})}{e^{-bh_0}(1-V_1V_2e^{2bh_0})}\times H_0^{(1)}(k_0r\cos\alpha)\cos\alpha d\alpha$$

dove $b=ik_0\sin\alpha$ e il cammino d'integrazione Γ è

$$\pi-i\infty\to\pi\to0\to-i\infty.$$

Deformando questo cammino, in modo da portarlo verso la retta $-i\infty$, i residui dei poli α_l dell'integrando (radici di $1-V_1V_2\exp 2bh_0=0$) danno luogo ai termini di uno spettro discreto, che per $k_0r\gg1$ è

$$\psi_{\text{discr.}}=\left\{\frac{\pi}{2k_0rh_0^2}\right\}^{\frac{1}{2}}e^{i\pi/4}\sum_{l=1}^{\infty}[e^{-bz}+V_1(\alpha_l)e^{bz}][e^{-bz_0}+V_1(\alpha_l)e^{bz_0}]\times\frac{\exp(ik_0r\cos\alpha_l)}{V_1(\alpha_l)\cos^{\frac{1}{2}}\alpha_l}.$$

Inoltre, poichè l'integrando possiede due punti di diramazione, rimangono anche due integrali lungo gli orli di due tagli che da questi punti vanno verso la retta $-i\infty$. Questi due integrali, che possono essere valutati col metodo del colle, rappresentano onde oblique, che all'infinito si comportano come $1/r^2$. *G. Toraldo di Francia* (Firenze).

Brehovskih, L. M. On the field of a point radiator in a stratified-inhomogeneous medium. I. Integral form of the solution. Izvestiya Akad. Nauk SSSR. Ser. Fiz. **13**, 505–514 (1949). (Russian)

The medium is stratified horizontally, the parameters depending only on a vertical coordinate z. There is a central layer which is homogeneous, and the parameters tend to definite limits as $z\to-\infty$, $z\to+\infty$; these limiting values are characterised, relative to the values for the central layer, by ratios n_1, n_2. The radiator is situated in the central layer and an expression is found for the field at points in this layer. Two cases are considered concurrently, that of a vertical dipole in a medium of unit permeability, and that of a point source of sound. The solution is obtained by Weyl's method of resolution into plane waves [Ann. Physik (4) **60**(365), 481–500 (1919)]. The reflection coefficients were obtained by the author in a previous paper [Akad. Nauk SSSR. Žurnal. Tehn. Fiz. **19**, 828–832 (1949)], and depend on the solution of a general Riccati equation. The final result gives the potential as a contour integral in terms of Hankel and exponential functions and the reflection coefficients. *F. V. Atkinson* (Ibadan).

Brehovskih, L. M. On the field of a point radiator in a stratified-inhomogeneous medium. II. Discussion of the solution. Izvestiya Akad. Nauk SSSR. Ser. Fiz. 13, 515–533 (1949). (Russian)

In the integral representation [see the preceding review] the contour is varied to yield two loop integrals, constituting a "continuous spectrum," and an infinite set of residues, or "discrete spectrum." Approximation to the loop integrals yields two "lateral waves," whose dependence on r, the distance, has the form $r^{-2}e^{ikr}$, propagated with velocities appropriate to the distant parts of the upper and lower half-spaces. The dependence of the waves on z is exponential-harmonic if neither n_1 nor n_2 is unity, but otherwise may be linear. It is assumed here that r is sufficiently large, precise criteria, too complicated to quote, being given. The waves of the discrete spectrum are, approximately, of the form $r^{-\frac{1}{2}}\exp(ik_0r-\frac{1}{2}ik_0r\alpha_l^2\pm ihlz/h)$, for a certain h and k_0, and $l=1, 2, \cdots$; asymptotic formulae for the constants α_l are given. Detailed results are given for two special cases, that in which the upper and lower half-spaces are homogeneous, and that of the "wave-guide" formed by two infinite parallel plates. In supplements additional formulae for the reflection coefficients are given, and it is proved, under additional assumptions, that the poles forming the discrete spectrum have no finite limit-point, the latter being inserted in response to criticisms made in a discussion of the author's work [see the following review]. Numerous approximations are made in the paper, the implied errors receiving only a cursory investigation.

F. V. Atkinson (Ibadan.).

Brehovskih, L. M. On the field of a point radiator in a stratified-inhomogeneous medium. III. Mean attenuation formulae. Izvestiya Akad. Nauk SSSR. Ser. Fiz. 13, 534–545 (1949). (Russian)

[Cf. the two preceding reviews.] The author states that in many cases there is a range of values of r for which it is necessary to take into account a large number of waves of the discrete spectrum, and for which the "lateral waves" are negligible. In view of attendant computational difficulties he derives mean-value formulae, wherein for given r the squared amplitude arising from the discrete spectrum is averaged over z, the height of the receiver, and z_0, the height of the sender, for positions within the central layer. For the acoustic case, if n_1 and n_2 are not both unity, the mean-square amplitude has an approximate expression of the form $Ar^{-1}\sum_{l=1}^{\infty}\exp(-Bl^2r)$, for certain constants A, B, while if $n_1=n_2=1$, the expression becomes $Ar^{-1}\sum_{l=1}^{\infty}\exp(-Bl^2r)$. For small r these expressions are of order r^{-1}, $r^{-\frac{1}{2}}$. For the electromagnetic case corresponding expressions are derived. The mean-square estimates of E_r, E_z and H_ϕ are, for small r, of orders r^{-2}, $r^{-\frac{1}{2}}$ and $r^{-\frac{1}{2}}$, respectively. However r must be large compared to the thickness of the central layer. These results are verified by the methods of geometrical optics.

Pages 546–548 contain a report of a discussion following a lecture by the author on the subject of the three papers under review. In the course of this the author expresses the view that his method would apply to the case of a horizontal dipole. He also promises a more detailed investigation of the poles forming the discrete spectrum.

F. V. Atkinson (Ibadan).

***Le Corbeiller, P. Matrix Analysis of Electric Networks.** Harvard Monographs in Applied Science, Number 1. Harvard University Press, Cambridge, Mass.; John Wiley & Sons, Inc., New York, N. Y., 1950. xi+112 pp. $3.00.

The author has presented with clarity, and in a very readable manner, Kron's method of network analysis. It seems to the reviewer that the author offers more enthusiasm than hard fact to support the claim that the method advocated has important practical advantages. No one will deny that the Kron method is a systematic procedure for reducing to routine computation the problem of finding the branch currents of an electrical network. But the implication that the amount of labor is thereby shortened is not substantiated. For, while the solution sought may formally be stated with admirable conciseness, numerical results require the computation of the inverse of a matrix, a matter not to be treated lightly. However, this book is heartily recommended to anyone dealing with electric circuits. Chapter I is an excellent introduction to the matrix analysis of networks and, although not complete, should be read by anyone contemplating serious work in network theory.

R. Kahal (Brooklyn, N. Y.).

***Duffin, R. J. Nonlinear electrical networks.** Proc. Symposia Appl. Math. 2, 66–70 (1950). $3.00.

A brief account of results previously presented [Bull. Amer. Math. Soc. 52, 833–838 (1946); 53, 963–971 (1947); 54, 119–127 (1948); 55, 119–129 (1949); these Rev. 8, 244; 9, 285, 503; 10, 536] and some related new results.

N. Levinson (Cambridge, Mass.).

***Wallman, Henry. Transient response and the central limit theorem of probability.** Proc. Symposia Appl. Math. 2, 91 (1950). $3.00.

The central limit theorem yields the result that "The step-function response of cascaded networks, each free of overshoot, tends to the error-function integral." This has important practical consequences, among them the fact that if a network is free of overshoot its time-of-response inevitably increases rapidly upon cascading, namely as the square-root of the number of cascaded networks. The underlying mathematical situation is the following: both the central limit theorem of probability and the theorem on cascaded step-function responses are consequences of the same theorem on iterated convolutions of bounded monotonic functions. *Extract from the paper.*

Gross, Hans-Georg. Die Berechnung der Stromverteilung in zylindrischen Leitern mit rechteckigem und elliptischem Querschnitt. Arch. Elektrotechnik 34, 241–268 (1940).

An integral equation for the current density over the cross-section of an infinitely long conducting cylinder, already given by Manneback [J. Math. Physics 1, 123–146 (1922)], is solved by a method of successive approximations applicable to any shape of cross-section. It is assumed that the frequency is so low that displacement currents may be neglected, and that the boundary of the cross-section is symmetric with respect to the x and y axes. Then the magnetic field must vanish at the origin and an expansion for the current density may be written in the form

$$j(x, y)=\sum j_\nu;\quad j_\nu=b_{2\nu}\rho^{2\nu}+\sum_{N=1}^{\nu-1}\rho^{2N}\sum_{\lambda=1}^{\infty}a_{2\lambda}^{(N)}\rho^{2\lambda}\cos 2\lambda\varphi,$$

where the first approximation is $j_0=$ constant; ρ, φ are cylin-

drical coordinates; and the coefficients b and a are expressed as integrals over the cross-section. The b terms correspond to the solution for a circular cylinder and lead to the known Bessel function solution. For an elliptic cylinder the νth component Fourier series terminates after ν terms while for a rectangular cylinder all the component series are infinite. Numerical values for the coefficients of the first five component series are tabulated for a cylinder of square cross-section and unit semi-diameter, also for an elliptic cylinder whose major and minor semi-axes are 1 and $\frac{1}{2}$, respectively. *M. C. Gray* (Murray Hill, N. J.).

Bouwkamp, C. J. On the effective length of a linear transmitting antenna. Philips Research Rep. **4**, 179–188 (1949).

The author suggests a more general definition of the effective length of a short linear antenna than that given by R. King [Proc. I.R.E. **29**, 458–463 (1941)]. Any assumed current distribution in the antenna may be represented as the sum of a symmetric and an antisymmetric distribution, and each component may be treated separately. The corresponding effective lengths are defined by the formulas

$$2h_sI_0=\int_{-h}^{h}I_s(z)[2J_1(\beta z)/\beta z]dz,$$

$$2h_aI_0=\int_{-h}^{h}I_a(z)[4J_2(\beta z)/\beta z]dz,$$

where I_0 is any assigned reference current level. The radiation resistance in each case can be expressed as the product of a numerical factor and the square of the effective length, and the total radiation resistance is the sum of the two components. The case of a top-loaded antenna, in which the current distribution is $I(z)=I_0\sin\beta(h+z)$, is worked out in detail. *M. C. Gray* (Murray Hill, N. J.).

Hartree, D. R. Some calculations of transients in an electronic valve. Appl. Sci. Research B. **1**, 379–390 (1950).

In a plane grid ($x=0$) and plate ($x=a$) structure, electrons with energy eV_0 are injected at $x=0$ starting at $T=0$ with a given current density time function $j(T)$; in particular, the step $j(T)=\text{constant}=J$ for $T>0$. A steady potential V_a is maintained at $x=a$. It is shown for the case $V_a=V_0$, from qualitative considerations based on Poisson's equation, that the initially injected electrons will in general arrive with energies greater than eV_0. The more general problem $V_a\neq V_0$ is then treated. It leads to an integro-differential equation involving the position coordinate $x(t, T)$ of the electron sheet injected at time T, at current time t:

$$d^2\xi(z, Z)/dz^2=U-Z+\int_0^z\xi(z, Z')dZ';$$

$$\xi=0,\quad d\xi/dz=\xi_0'\text{ at }z=Z,$$

where z, Z, ξ; U are non-dimensionals in lieu of t, T, x; V_a-V_0, for the step function case. The corresponding equation for cylindrical symmetry is nonlinear. A closed solution is obtained via power expansion in terms of Airy functions and their integrals. Numerical tabulation is given. The solution must be analytically continued between domains through singularities corresponding to instances of incipient overcrossing of electrons injected at different times: $\partial\xi/\partial z=0$ and of arrival at $x=a$ or back-arrival at $x=0$. A numerical example is treated with graphs presented for three consecutive domains of this type, which approaches the steady rate. *H. G. Baerwald* (Cleveland, Ohio).

Kahan, T., et Eckart, G. La fonction de Green des ondes électromagnétiques dans un demi-espace à discontinuité plane. J. Phys. Radium (8) **10**, 333–341 (1949).

The authors' aim is to get a clear insight into the behaviour of a duct along the surface of the earth in relation to ultra-short-wave propagation. To that end they assume a perfectly conducting plane earth and a two-layer atmosphere, the interface between the two layers being parallel to the earth's surface. The lower layer has the greater dielectric constant. A point source (vertical magnetic dipole) is assumed at an arbitrary height in the lower layer. The field is derived from a scalar wave function which is expressed in terms of an infinite integral involving the zero-order Bessel function, as in the conventional approach to propagation problems in the presence of a plane earth. The transformation and evaluation of this integral, its asymptotic expansion, etc., are to be treated in a later paper. The physical interpretation of the mathematical results is discussed in detail. *C. J. Bouwkamp* (Eindhoven).

Kahan, Théo. Application de la méthode des approximations successives de Picard à l'étude des discontinuités dans les guides d'ondes. C. R. Acad. Sci. Paris **230**, 527–529 (1950).

If a circular wave guide is bent according to some geometric law, under certain assumptions regarding the bend it is possible to give a perturbation solution of the Maxwell equations. This method of approximation is due to Picard and has been utilized by Schelkunoff. *A. E. Heins.*

Ellis, Wade. Theory of metal-plate prisms for microwaves. Air Force Cambridge Research Laboratories, E 5052, iv+20 pp. (1949).

The author discusses the behaviour of normal and binormal metal-plate prisms for microwaves from the point of view of geometrical optics. A normal prism is defined as a prism in which the plates are evenly spaced, congruent, isoceles triangles. In a binormal prism the plates are evenly spaced rectangles (parallel to the bottom) of constant length and gradually diminishing width. For either type the deviation and dispersion of waves are calculated on the assumption that the incident ray is normal to the edge of the prism and the electric vector parallel to the plates. Whereas the normal prism has an angle of minimum deviation (as the ordinary optical prism), the binormal type is shown to have an angle of maximum deviation. *C. Bouwkamp.*

Clemens, George J. A tapered line termination at microwaves. Quart. Appl. Math. **7**, 425–432 (1950).

The author approximates an experimental type of transmission line (with a linear taper) by a line in which the variation (in the longitudinal direction, x) of the outer conductor is given by $r_0=a(b/a)^{\exp(-kx)}$ [instead of the author's equation (4)], where a and b are the radii of the inner and outer conductors far away from the taper. The corresponding differential equation for the current along the line is solved in terms of Bessel functions. An alternative solution is derived by means of an integral equation, following a perturbation method suggested by Schelkunoff. [This integral equation might have been obtained also from the differential equation by the method of variation of constants.] A discussion of the differences between the calculated and measured values of standing-wave-ratios is included. *C. Bouwkamp* (Eindhoven).

Wise, W. Howard. Capacity of a pair of insulated wires. Quart. Appl. Math. **7**, 432–436 (1950).

Formulas and methods for computing capacity from the results found by J. W. Craggs and C. J. Tranter [Quart. J. Math., Oxford Ser. (1) **17**, 138–144 (1946); these Rev. **8**, 298] are presented and discussed. *R. V. Churchill.*

Abelès, Florin. Deux théorèmes relatifs aux couches minces et leurs applications. C. R. Acad. Sci. Paris **230**, 632–633 (1950).

Mercier, André. Sur les fondaments de l'électrodynamique classique (méthode axiomatique). Arch. Sci. Soc. Phys. Hist. Nat. Genève **2**, 584–588 (1949).

The author states some axioms regarding the quantities entering in the discussion of the electromagnetic field in a dielectric medium. *A. H. Taub* (Urbana, Ill.).

Hoffmann, Banesh. Kron's non-Riemannian electrodynamics. Rev. Modern Physics **21**, 535–540 (1949).

The motion of a dynamical system having n degrees of freedom may be represented by that of a point having coordinates x^s in an n-dimensional Riemannian space with metric L_{st}. The path is given by the dynamical equation

$$L_{su}\ddot{x}^u + \Gamma^t_{su,r}\dot{x}^r\dot{x}^t + R_{st}\dot{x}^t = F_s,$$

the comma denoting the covariant derivative formed from the metric L_{st}; R_{st} is the dissipation tensor, F_s are the generalized impressed forces. For certain electrodynamical systems the kinetic energy has the form

$$T = \tfrac{1}{2}L_{ab}\dot{x}^a\dot{x}^b + \tfrac{1}{2}L_{st}\dot{x}^s\dot{x}^t,$$

where a, b refer to electrical degrees of freedom and s, t to mechanical. The behavior of an electrodynamical system may be represented by the motion of a point x^u in an $(m+n)$-dimensional Riemannian space in which m of the coordinates obey a cylinder condition (permanent absence of specific coordinates from the components of all geometric objects). In the theory of rotating electrical machines one has occasion to introduce a non-holonomic reference system. The transformation leading to this non-holonomic reference frame is of the form

$$d\bar{x}^b = c_u{}^b(x^s)dx^u, \quad \bar{x}^s = \bar{x}^s(x^t),$$

in which the mechanical coordinates transform holonomically while the electrical differentials transform by means of a matrix depending only on the mechanical coordinates. The author regards all geometric objects in the new non-holonomic reference frame as belonging to a holonomic system, say with coordinates $\bar{x}^\lambda$, in a new space differing from the original Riemannian space of x^u. The new space can be regarded either as a Riemannian space referred to a non-holonomic reference frame or else as a non-Riemannian metric space, with torsion, referred to a holonomic reference frame. The author discusses the simple case of a single phase squirrel cage induction motor. *M. Pinl* (Dacca).

Caprioli, Luigi. Sul campo elettromagnetico in un semispazio. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 103–106 (1949).

For the electromagnetic field in a semi-infinite conductor of (complex) wave number k the author obtains the formula

$$\mathbf{E}(P) = \int_\sigma 2[\operatorname{grad} r_1 \wedge (\mathbf{E}_t \wedge \mathbf{n})]\frac{d}{dr_1}\left\{\frac{1}{4\pi r_1}e^{-ikr_1}\right\}d\sigma$$

which determines the electric field inside the conductor once the tangential component of the field on the boundary is known. In the derivation he uses some results of a paper in the course of publication in which a similar problem is discussed for a conductor of any shape. Here P is any interior point of the solid, σ the plane boundary, N any point of σ, $\mathbf{n}$ the exterior normal and $d\sigma$ the surface element at N, r_1 is the distance PN, $\mathbf{E}$ the electric field vector and $\mathbf{E}_t$ its tangential component. The formula can be applied to a problem of radiotelegraphy. *A. Erdélyi.*

Truell, Rohn. A method of measuring the field strength of high-frequency electromagnetic fields. Proc. I.R.E. **36**, 1249–1251 (1948).

The case of an electron beam, velocity v_z, in a longitudinal homogeneous magnetic d.c. field H_z and transversal electric a.c. field of amplitude E_x of cyclotron frequency ω is analyzed. For the simplified case of homogeneous E_x over a finite length l and $v_z \ll l\omega$, the beam would simply spread out into a circle of radius $r = cE_x/(2v_zH_z)$ if H_z were chosen such that ω became equal to the Larmor frequency $\omega_c = eH_z/(mc)$. This permits one to measure E_x by means of an intercepting circular aperture in front of an electron collector according to $E_x \simeq 320 r V_z^{\frac{1}{2}}$ (volts/cm), V_z = accelerating voltage. If E_x is the field in the center plane of a plane parallel condenser of dimensions $\ll$ half wavelength, this analysis is in error due to fringe fields. These are taken into account approximately by a Schwartz-Christoffel mapping yielding a length correction of the order d/l = spacing. *H. G. Baerwald.*

Truell, Rohn. Problems related to measuring the field strength of high-frequency electromagnetic fields. Proc. I.R.E. **37**, 1144–1147 (1949).

The analysis presented in the preceding paper is improved by taking into account: (1) the electron arrival phases; (2) the actual field distribution $E_x(z)$ near the beam (center) axis, i.e., for the particular case of a rectangular cavity resonator in its basic mode. The analysis shows that: (1) circular beam spreading obtains strictly only for the specific values of injection velocity $v_z = \omega l/(2n\pi)$, $n = 1, 2, \cdots$, or $\omega_c\tau = 2n\pi$, τ the transit time, this condition becoming noncritical for $n \gg 1$, $n = O(10^3)$, say; (2) the critical value of the cyclotron frequency ω is no longer $\omega_c = eH_z/(mc)$, but either of the "doublet" components $\omega_c = \pm\pi v_z/l$. Also, $E_x = 4v_zH_x/(lc)$ for a sinusoidal field, which is twice the value obtained for the earlier approximate case of a homogeneous field.

H. G. Baerwald (Cleveland, Ohio).

***Truell, Rohn. Problems related to measuring the field strength of high frequency electromagnetic fields.** Proc. Symposia Appl. Math. **2**, 85 (1950). $3.00.

Abstract of the paper reviewed above.

Eliezer, C. Jayaratnam. A note on electron theory. Proc. Cambridge Philos. Soc. **46**, 199–201 (1950).

As a model of the classical electron which shows promise of not giving self-accelerating solutions a sphere of radius a with charge confined to its surface is treated nonrelativistically, the entire mass m being supposed of electromagnetic origin. For a free electron the motion is self-oscillatory with period equal to the time taken by light to go a distance $2a$. In a uniform electric field E there is superposed on this motion a uniform acceleration eE/m. For an electric field of period $2\pi/\omega$ there are resonance effects for $a\omega = n\pi$, (n an integer). *C. Strachan* (Aberdeen).

Schumann, W. O. Wellen längs homogener Plasmaschichten. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1948**, 255–279 (1949).

Die Arbeit beschäftigt sich mit der Wellenausbreitung längs Plasmaschichten, wobei als angrenzende Halbräume Luft, metallische Leiter und deren Kombinationen betrachtet werden. Als Plasma wird eine ladungskompensierte Gasschicht angenommen, in der freie Elektronen vorhanden sind, welche eine zusätzliche frequenzabhängige Dielektrizitätskonstante $\Delta' = Ne^2/\omega^2 m$ verursachen. Aus den bekannten Stetigkeitsbedingungen an den Grenzflächen erhält man die Zusammenhänge zwischen den in den bei allen ebenen Problemen mit Hilfe von trigonometrischen und Exponentialfunktionen ausdrückbaren Lösungen enthaltenen Konstanten. Als erstes Problem wird eine Plasmaschicht zwischen zwei Luftschichten behandelt. Bei zunehmender Dicke tritt im Plasma ein Skineffekt auf und mit zunehmender Frequenz auch in der angrenzenden Luftschicht. Im Falle eines Plasmazylinders in Luft werden die Feldstärken mit Hilfe von der Besselschen Funktion erster Art J_0, der entsprechenden Hankelschen Funktion H_0 und deren ersten Ableitungen ausgedrückt. Die erhaltenen Resultate sind übrigens denen im ersten Beispiel hergeleiteten sehr ähnlich, nur nimmt die Feldstärke im angrenzenden Luftraum schneller ab. Als weitere Fälle werden eine Luftschicht zwischen zwei Plasmahalbräumen, ebenfalls eine Luftschicht zwischen einem Leiter und einem Plasmahalbraum und eine Plasmaschicht zwischen zwei leitenden Halbräumen besprochen.

T. Neugebauer (Budapest).

Novobátzky, K. F. Bewegtes Dielektrikum. Hungarica Acta Physica **1**, 25–34 (1949).

The author presents a classical phenomenological theory of electromagnetism in a moving dielectric medium. This is based on a Lagrangian which differs from that used by Jauch and Watson [Physical Rev. (2) **74**, 950–957 (1948); these Rev. **10**, 346] and has the property that the derived energy density is positive definite. The author then argues that Jauch and Watson have misinterpreted their results when they state that negative energy photons occur for a medium velocity greater than c/n, where n is the refractive index.

A. H. Taub (Urbana, Ill.).

Johnson, M. H., and Lippmann, B. A. Relativistic motion in a magnetic field. Physical Rev. (2) **77**, 702–705 (1950).

A simple integral of the Dirac equations for a charge in a magnetic field is used to reduce these equations to two simultaneous first order partial differential equations. The classification of the solutions of these equations is discussed.

A. H. Taub (Urbana, Ill.).

Quantum Mechanics

Pirenne, Jean. Note sur le formalisme des champs quantifiés. Physica **15**, 1023–1031 (1949).

This note provides a proof of the commutation rules for the field operators in quantum field theory, clearly displaying their relation to the statistics of the particles considered. The representations of a system of similar particles by $\Phi(n; x, t)$ in configuration space or by $\Psi(n_1, n_2, \cdots; t)$ in quantum space are connected with each other by a unitary operator S. The author shows that S can be expressed in terms of the semi-field operators. Under a symmetric permutation of the particles Ψ is invariant, whereas Φ is invariant or alternates in sign according as the statistics are Bose-Einstein or Fermi-Dirac. This difference in behaviour under the permutation group is reflected in S, requiring the known commutation rules for the field operators. An obvious advantage of the present treatment over the usual ones is the fact that the commutation rules for the different types of statistics are treated simultaneously and appear equally natural. [Misprints which are troubling: the inversion of the subscript on the commutator in equation 34b and the interchange of C_1 and C_2 immediately following.]

A. J. Coleman (Toronto, Ont.).

Yukawa, Hideki. Quantum theory of non-local fields. Part I. Free fields. Physical Rev. (2) **77**, 219–226 (1950).

This paper is a continuation and extension of a previous article by the same author [same Rev. (2) **76**, 300–301 (1949)]. A "non-local" field is characterized by operators which are not point functions in ordinary space. In his introduction, Yukawa points up the defects of current tricks for avoiding infinities in quantum field theories. Basically they consist in introducing rather artificial convergent combinations of divergent fields. Yukawa asserts that it would clearly be preferable to have convergent irreducible fields. He seeks such among nonlocal fields. The case treated in considerable detail in the present paper is that of a relativistically invariant nonlocal scalar field. This is characterized by a single scalar operator $U(x^\mu, p_\mu)$ which differs from the analogous operator in local field theory by depending on p_μ (the momenta operators) as well as x^μ (the position operators). He assumes the equations $[x^\mu, p_\nu] = i\hbar\delta_{\mu\nu}$, $[p_\mu[p^\mu, U]] + m^2c^2U = 0$, of ordinary quantum field theory, but because of the dependence of U on the p_μ these are insufficient, and he adopts the additional equations $[x_\mu[x^\mu, U]] - \lambda^2 U = 0$, $[p_\mu[x^\mu, U]] = 0$. The first of these is characteristic of Yukawa's theory and is "justified" by an appeal to the principle of reciprocity which has been discussed frequently by Born [e.g., Rev. Modern Physics **21**, 463–473 (1949)]. The constant λ has the dimensions of length and is interpreted as the radius of the elementary particle associated with the field in question. By use of second quantization Yukawa shows that, for $m \neq 0$, the nonlocal scalar field corresponds to an assembly of bosons (both charged and uncharged are discussed) of mass m, radius λ, spin 0, in states characterized by a linear momentum quantum number and two new quantum numbers which may be interpreted as specifying an internal rotation. For $m = 0$, the constant λ is not susceptible of so easy an interpretation. Detailed discussion of this case is promised in a later paper.

The author asserts that all the above carries over without essential alteration to nonlocal vector fields. Spinor fields, however, must be treated differently and these are briefly discussed. It is shown that relativistically invariant equations can be found for a spinor field which, it is asserted (details of the second quantization are not given but promised in a subsequent article), represents an assembly of fermions of mass m, radius λ and spin $\frac{1}{2}$. The paper concludes with the conjecture that in extending the above to interacting nonlocal fields it will be impossible to use differential equations and recourse will have to be made to an integral formalism such as Heisenberg's S-matrix.

A. J. Coleman (Toronto, Ont.).

Mercier, André. Sur le formalisme canonique, la condition accessoire $H+p_{f+1}=0$ et la réversibilité de la mécanique classique et de la théorie des quanta. Arch. Sci. Soc. Phys. Hist. Nat. Genève **2**, 403–422 (1949).

The author first reviews the behavior of the Poisson brackets of classical mechanics for a system with f degrees of freedom under the transformations $t\to -t$, $p_k\to -p_k$, $q_k\to q_k$ and $t\to -t$, $p_k\to p_k$, $q_k\to -q_k$. He then introduces $f+1$ variables q_k, t; p_k, p_k+1 and shows that the quantity $\mathcal{H}=H+p_{f+1}$ is a first integral of the equations of motion. This is the so-called homogeneous formalism. The paper concludes with a correspondence principle argument for the derivation of the Schrödinger equation from the homogeneous formalism. *A. H. Taub* (Urbana, Ill.).

Green, H. S. On the self-energies and cross-sections of orthodox quantum mechanics. Proc. Roy. Soc. London. Ser. A. **197**, 73–89 (1949).

The principal object of this paper is to decide to what extent the infinites which appear in pre-Schwingerian quantum mechanics arise from inadequacies in the mathematical technique of the perturbation methods used rather than from basic theoretical assumptions. However, perhaps of greater importance are two subsidiary elements of the paper: (a) a new perturbation method, (b) the development of field theory in terms of the density operator. The idea of using a density operator rather than a wave vector to describe a system or ensemble is not at all new [cf. J. von Neumann, Mathematische Grundlagen der Quantenmechanik, Springer, Berlin, 1932, chapter IV]. Born and Green have recently used it in a variety of contexts. Here Green begins to develop field theory in terms of the density operator, and shows that the symmetrization of the energy-momentum tensor and the development of second quantization are greatly simplified thereby. In particular, negative expectation values for energy make an almost natural appearance.

The usual perturbation methods of quantum mechanics become almost intractable for problems with a high degree of degeneracy and, in general, give rise to infinite series whose convergence it is difficult, if not impossible to determine. Green's new perturbation method always leads to a convergent solution when such exists. When the perturbing energy is independent of the time, the functions $\delta\pm(E-E')$ used by Heisenberg [Z. Physik **120**, 513–538, 673–702 (1943); these Rev. **4**, 292] play a natural and fundamental role in Green's method. In an appendix the author gives a simple example of a problem with finite eigenvalues for which the series provided by the usual perturbation method sometimes diverges, and even when it converges is not as good in the fifth approximation as the first approximation given by his method.

Green applies the above to collision cross-sections and self-energy problems. He obtains formulas for cross-sections which are necessarily convergent. The convergence of his formula for self-energy depends on the particular problem under consideration. He applies it to the interaction of the electron and photon fields and finds that orthodox theory, even with his improved perturbation technic, leads to divergent transverse self-energy. In an appendix, a process for the symmetrization of the energy-momentum tensor is formulated which is essentially a generalization of Appendix I of Wentzel's "Quantum Theory of Fields" [Interscience Publishers, New York, 1949; these Rev. **11**, 146] to the case where the Lagrangian can involve arbitrarily high derivatives of the field variables. Green's formulas become identical to those given independently by J. van Isacker [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 451–456 (1949); these Rev. **11**, 146] when the L, $L^{\alpha\beta}$, $L^{\alpha\beta,\gamma}$, $T^{\alpha\beta}$ of Green are replaced by M, $-2T^{\alpha\beta}$, $S^{[\alpha\beta]\gamma}$, $-\vartheta^{\alpha\beta}$ of van Isacker.

A. J. Coleman (Toronto).

Bauer, Hans, und Jensen, J. Hans D. Zur Strahlungsdämpfung des Oszillators. Z. Physik **124**, 580–585 (1948).

Schrödinger showed that a wave-packet consisting of eigenfunctions of a harmonic oscillator persists, its center of gravity oscillating classically. The authors show that, during radiation with change of quantum number, the packet likewise persists, its center of gravity undergoing damped harmonic motion. This was previously shown by F. Bloch; the authors simplify his calculations, using a method of Weisskopf and Wigner. *O. Frink* (State College, Pa.).

Goudot, Mme. Andrée. Sur la mécanique ondulatoire du corps solide rigide. C. R. Acad. Sci. Paris **229**, 646–647 (1949).

The author derives formulas for the motion of a rigid body about a fixed point, using the Eulerian angles, from the point of view of the quantum kinematics of J. Viard.

O. Frink (State College, Pa.).

Goudot, Mme. Andrée. Propriétés cinématiques du corps rigide en mécanique ondulatoire. C. R. Acad. Sci. Paris **230**, 1140–1142 (1950).

Faure, Robert. Méthodes d'intégration communes à la mécanique classique et à la mécanique ondulatoire. C. R. Acad. Sci. Paris **230**, 364–365 (1950).

Dem Übergang von der Wellenoptik zur geometrischen Optik entspricht bekanntlich der Übergang von der Wellengleichung zweiter Ordnung zur Hamilton-Jacobischen Differentialgleichung erster Ordnung. Diese Parallele lässt sich verallgemeinern. Verf. gewinnt als notwendige Bedingung, Integrationsresultate der klassischen Mechanik zur Bestimmung von Eigenfunktionen der Wellenmechanik ausnützen zu können, die Existenz eines ersten Integrals für das klassische Problem. *M. Pinl* (Dacca).

Géhéniau, Jules. Étude sur les champs spinoriels et leur quantification. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **21**, no. 5, 59 pp. (1948).

The author reviews that part of the theory of four-component spinors needed for a discussion of the Dirac equation for simple spinors and the generalisation of this equation to spinors with n indices. The known results regarding the spin and mass of particles whose wave functions satisfy such equations are stated. There is also included a review of the second quantization of such spinor fields.

A. H. Taub (Urbana, Ill.).

Ferretti, B. Sulla diagonalizzazione della hamiltoniana nella teoria dei campi d'onda. Nuovo Cimento (9) **7**, 79–81 (1950).

Dyson's [Physical Rev. (2) **75**, 486–502, 1736–1755 (1949); these Rev. **10**, 418; **11**, 145] discussion of the Tomonaga-Schwinger-Feynman formalism is supplemented by the remark that under certain conditions his S-operator can be considered as the generating operator of a canonical transformation diagonalizing the Hamiltonian of the wave fields in interaction. To show this, the assumption is made

that the Hamiltonian of the system can be represented in the form $H=H_0+\lambda(t)H'$, where H' is the perturbation and $\lambda(t)$ an "adiabatic switch-on" function approaching unity for finite values of t. The operator $H_T=S^{-1}(H_0+\underline{H}')S$, with $\underline{H}'=\exp(iH_0t)H'\cdot\exp(-iH_0t)$, $(\hbar=1, c=1)$, then turns out to be diagonal, provided that H' does not contain any derivatives of the field operators. If it were possible to generalize this result to the case of discrete eigenvalues, it would be of more than merely formal interest. *E. Gora.*

Stueckelberg, E. C. G., et Rivier, D. Causalité et structure de la matrice *S*. Helvetica Phys. Acta **23**, 215–222 (1950).

In earlier papers [e.g., Rivier, Helvetica Phys. Acta **22**, 265–318 (1949); these Rev. **11**, 301] it has been shown that the development of the S-matrix should contain only one particular combination of the so called D-functions (that is, solutions of $(\Box-\varkappa^2)D=-\delta(x)$, where $\delta(x)$ is Dirac's delta-function), describing outgoing waves for the future and incoming waves for the past, and excluding negative energy states. Other possibilities have to be excluded as incompatible with the principle of causality. This "causality condition" is now generalized to an arbitrary number of variables, and shown to be useful for determining the form of higher order terms in series expansions for the S-matrix. If a Hamiltonian formalism is used as starting point, usually such higher order terms contain divergent parts which have to be eliminated by some subtraction procedure. The authors propose a different method: to start with the convergent first order approximation and to adapt the S-matrix expansion to it in such a way as to fulfil the above "causality condition." "Causal corrections" are thus added to the higher order terms. This method is obviously ambiguous, but the authors intend to show that it leads to second order corrections analogous to those of Schwinger for the polarization of the vacuum and the self-energies of elementary particles. *E. Gora* (Providence, R. I.).

Katayama, Y. An attempt to Pauli's regulator. Progress Theoret. Physics **4**, 372–374 (1949).

The author proposes to base the regulation method of Pauli and Villars [Rev. Modern Physics **21**, 434–444 (1949); these Rev. **11**, 301] upon a formalism in which electromagnetic and electron-positron fields satisfy wave-equations of higher order than the usual Maxwell and Dirac equations. The generalised fields satisfying the higher-order equations can be analysed into a combination of conventional wave-fields of different masses [B. Podolsky and P. Schwed, Rev. Modern Physics **20**, 40–50 (1948); these Rev. **9**, 551], and so give results identical to those of Pauli. The author does not claim that his method is more realistic than that of Pauli; in both methods the same obstacles prevent a consistent physical interpretation being given to the formalism. *F. J. Dyson* (Princeton, N. J.).

Tonnelat, Marie-Antoinette. Résolution des équations fondamentales d'une théorie unitaire purement affine. C. R. Acad. Sci. Paris **230**, 182–184 (1950).

The author describes the algebraic steps for solving the Euler equations of a variational principle, previously described [same C. R. **228**, 368–370 (1949); these Rev. **10**, 408], for the affine connection. *A. H. Taub.*

Valatin, Jean G. Le couplage des variables de spineur d'un système d'électrons de Dirac. J. Phys. Radium (8) **11**, 97–101 (1950).

Valatin, Jean G. Le transformé de charge des opérateurs de la théorie de l'électron de Dirac. C. R. Acad. Sci. Paris **230**, 822–823 (1950).

Valatin, Jean G. La correspondance entre les opérateurs de la théorie d'une seule particule et de la théorie du positron. C. R. Acad. Sci. Paris **230**, 925–927 (1950).

Valatin, Jean G. La représentation d'interaction et l'espace de configuration. C. R. Acad. Sci. Paris **230**, 1456–1458 (1950).

Päsler, Max. Behandlung des Raumrotators im Unterbereich der Laplace-Transformation. Ann. Physik (6) **6**, 365–374 (1949).

Schwinger, Julian. Quantum electrodynamics. III. The electromagnetic properties of the electron—radiative corrections to scattering. Physical Rev. (2) **76**, 790–817 (1949).

"The discussion of vacuum polarization in the previous paper of this series [same Rev. (2) **75**, 651–679 (1949); these Rev. **10**, 663] was confined to that produced by the field of a prescribed current distribution. We now consider the induction of current in the vacuum by an electron, which is a dynamical system and an entity indistinguishable from the particles associated with vacuum fluctuations. The correction to the current operator produced by the coupling with the electromagnetic field is developed in a power series, of which first and second order terms are retained. One obtains second-order modifications in the current operator which are of the same general nature as the previously treated vacuum polarization current, save for a contribution that has the form of a dipole current. The latter implies a fractional increase of $(\alpha/2\pi)$ in the spin magnetic moment of the electron.

"We consider in the second section the interaction of three systems, the matter field, the electromagnetic field, and a given current distribution. This situation can be described in terms of an external potential coupled to the current operator modified by the interaction with the vacuum electromagnetic field. Application is made to the scattering of an electron by an external field, in which the latter is regarded as a small perturbation. The correction to the cross section for radiationless scattering is determined by the second-order correction to the current operator, while scattering accompanied by single quantum emission is a consequence of the first-order current correction. The final object of calculation is the differential cross section for scattering through a given angle with a prescribed maximum energy loss, which is completely free of divergences. Detailed evaluations are given in two situations, the essentially elastic scattering of an electron, in which only a small fraction of the kinetic energy is radiated, and the scattering of a slow electron with unrestricted energy loss." From the author's summary.

This is the first published account explaining in detail the application of the author's theory to a specific observable phenomenon. The paper is full of mathematical devices which will be useful to any physicist who is undertaking similar calculations. The most important departure from the methods of the author's previous papers is that the calculations are now carried out in momentum-space, and the evaluation of momentum integrals is based on the

formula

$$\int\int\int\int dk_1 dk_2 dk_3 dk_0 \delta''(k_1^2+k_2^2+k_3^2-k_0^2+\lambda^2)=(\pi/\lambda^2),$$

where δ'' is the second derivative of the Dirac δ-function.

F. J. Dyson (Princeton, N. J.).

*Feshbach, Herman. **The new quantum electrodynamics.** Proc. Symposia Appl. Math. 2, 1–19 (1950). $3.00.

An expository article describing the development of electrodynamics from the classical theory of Lorentz to the covariant quantum theory of Schwinger. The ideas of the Schwinger theory are explained in some detail; the article can be recommended as an introduction to the published papers of Schwinger [Physical Rev. (2) 74, 1439–1461 (1948); 75, 651–679 (1949); these Rev. 10, 345, 663; the paper reviewed above]. *F. J. Dyson* (Princeton, N. J.).

Géhéniau, J., et Villars, F. La self-énergie de l'électron dans un champ électromagnétique extérieur. Helvetica Phys. Acta 23, 178–186 (1950).

Schwinger's method [Physical Rev. (2) 74, 1439–1461 (1948); 75, 651–679 (1949); these Rev. 10, 345, 663] of calculating the self-energy of a free electron is applied to a charged particle in an electromagnetic field. For the additional magnetic moment of the electron the result obtained is the same as Schwinger's in a more recent paper [see the 2d preceding review]. Further, the authors calculate the self-energy and the magnetic moment of the proton, assuming a pseudo-scalar meson field, and confirm Luttinger's [Helvetica Phys. Acta 21, 483–496 (1948)] and Case's [Physical Rev. (2) 76, 1–13 (1949)] results.

E. Gora (Providence, R. I.).

BIBLIOGRAPHICAL NOTES

*Berzolari, Luigi, editor. **Enciclopedia delle Matematiche Elementari. Volume Terzo. Parte Seconda.** Ulrico Hoepli, Milano, 1950. xix+1038 pp. (1 plate). 3800 Lire.

This is the final installment of a work of which the first part appeared in 1930. The contents are as follows: U. Cassina, Approssimazioni numeriche; F. Sibirani, Calcolo delle probabilità; C. Gini, Metodologia statistica: la misura dei fenomeni collettivi; T. Boggio, Matematica finanziaria; T. Boggio e F. Giaccardi, Matematica attuariale; M. Cipolla, Matematica recreativa; Ettore Bortolotti, Storia della matematica elementare; F. Severi e F. Conforto, Caratteri e indirizzi della matematica moderna; M. Gliozzi, Storia del pensiero fisico; L. Brusotti, Questioni didattiche; G. Giorgi, Sui fondamenti della geometria.

*Rajalakshman, D. V., editor. **Sixteenth Conference of the Indian Mathematical Society, Madras, 1949.** ix+102 pp.

Material of mathematical interest contained in this volume includes an article on recent mathematical work in Madras and a biographical sketch of Ramanujan.

Annales de l'Institut Fourier.

Volume 1 (for 1949) appeared in 1950. This is a continuation of Annales de l'Université de Grenoble (Nouvelle Série), Section des Sciences Mathématiques et Physiques, of which the last volume was 23.

Arkiv för Matematik, Astronomi och Fysik.

This publication has been replaced by four separate publications, **Arkiv för Astronomi** (Band 1, Häfte 1 issued in 1950); **Arkiv för Fysik** (Band 1, Häfte 1 issued in 1949); **Arkiv för Geofysik** (Band 1, Häfte 1 issued in 1950); **Arkiv för Matematik** (Band 1, Häfte 1 issued in 1949).

Bulletin de la Société Mathématique de Belgique.

The first volume (carrying no volume number), covering the years 1947–48, was issued in 1949.

Calcutta Statistical Association Bulletin.

Number 1 was issued in August, 1947. Volume 2 began with number 5.

Gaceta Matemática.

Volume 1, nos. 1–8, of series 1 appeared in 1949. This is a publication of the Instituto "Jorge Juan" de Matemáticas and the Real Sociedad Matemática Española.

Journal of the Osaka Institute of Science and Technology (Kinki University). Part I. Mathematics and Physics.

Volume 1, number 1 is dated May, 1949.

Matematikai Lapok.

Volume 1, number 1 is dated 1949. This is a continuation of Matematikai és Fizikai Lapok. Papers are in Hungarian with Russian and English summaries.

Mathematisch-Physikalische Semesterberichte.

Band 1, Heft 1/2 is dated 1949; the journal is published at Göttingen. "Diese Blätter dienen der mathematischen und physikalischen Weiterbildung auf den Hochschulen ausgebildeter Fachleute."

Mitteilungsblatt für Mathematische Statistik.

Volume 1, number 1 is dated April, 1949. The journal is published in Munich.

Nachrichten der Österreichischen Mathematischen Gesellschaft.

Volume 3, no. 8/9 (1949) contains abstracts of the papers presented at the II. Österreichische Mathematikerkongress, 1949.

Publicationes Mathematicae.

Volume 1, number 1, is dated 1949. This is a publication of the Mathematical Institute of the University of Debrecen. Papers are in English, French or German.

Publicationes Mathematicae Collectae Budapest.

Volume 1 was issued in 1949. It contains 27 papers, all but one of which have been published elsewhere. The title also appears in Hungarian, Matematikai Értekezésgyüjtemény Budapest. This is a publication of the Mathematical Institute of the University of Budapest.

Revista di Matematica della Università di Parma.

Volume 1, number 1 is dated January–February, 1950. Five numbers a year are planned.